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A METHOD OF DETERMINING THE RADIANT LUMINOUS EFFICIENCY OF A LIGHT SOURCE BY MEANS OF A CELL WHOSE TRANSMISSION CURVE IS IDENTICAL WITH THE LUMINOSITY CURVE OF THE AVERAGE EYE.[1]

By Enoch Karrer.

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CONSIDERABLE work has been done upon the subject included in the above title, under the names of "luminous efficiency" and "radiant efficiency." Very little of it, however, can have any bearing upon or connection with other photometric quantities, because in most instances no account was taken of the fact that all radiation in the visible spectrum is not equally powerful in producing light.

By the radiant luminous efficiency of any source is meant the ratio of the energy of the radiation emitted *weighted in accordance with its effectiveness* in producing the sensation of light, to the energy of the total radiation emitted.[2] This may be indicated thus:

$$\text{Radiant Luminous Efficiency} = \mu_R = \int_0^\infty \frac{E_\lambda (V_\lambda) d\lambda}{E_\lambda d\lambda},$$

[1] Abstract of this paper was read before the New York meeting of Physical Society October 31, 1914.

[2] For the definition of this quantity in relation to other suggested definitions of photometric quantities see H. E. Ives, Lighting Journal, Oct., 1913.

where V_λ is the visibility of wave-length λ obtained from the equal energy luminosity curve[1] of the normal eye; and $E_\lambda d\lambda$ is the energy at wave-length λ.

To see clearly what is meant by the luminosity mentioned above we may proceed as follows: Suppose we have a spectrum of a carbon filament lamp. If now we take from the region of wave-length .55 μ a certain amount of energy measured with a thermo-couple, say, and compare its brightness with the light of a candle, we will find a certain number representing its intensity, say one. Now, if energy is taken from some other region, say near wave-length .61 μ, we will find that for the same amount of energy we get approximately one half the candle power, or the number 0.5 representing its intensity. Thus we might go through the spectrum and determine the "light-value" of a definite and constant amount of energy at the various wave-lengths.

The equal energy luminosity of the observer's eye, then, is obtained by plotting the values determined above against the corresponding wave-lengths. This has been done by Ives[1] for many eyes; and a curve for the "average eye" has been obtained. This has a very important bearing upon many determinations and problems in photometry; and it is readily seen that the determination of luminous efficiencies made by merely selecting a certain region in the spectrum, from .4 μ to .72 μ as has been commonly done, without regard to the manner in which this radiation affects the eye, can have no accurate meaning in practice.

The object of the present research was to develop a method of measuring radiant luminous efficiencies according to the above considerations.

There are several possible modes of obtaining the quantity defined as the radiant luminous efficiency.

First. An energy curve of the source in question may be plotted. The ordinates of this curve represent the energy of the radiation at various wave-lengths (abscissæ). Now, if these ordinates be multiplied by the ordinates of the luminosity curve, representing the weight to be given each wave-length in producing light, we obtain numbers which will represent the light produced by each wave-length. A curve then may be drawn with these numbers as ordinates and wave-lengths as abscissæ. The area of this latter curve is proportional to the energy of the radiation properly rated as to its "light-value." The area included by the original energy curve measures the total energy of the radiation. The ratio of the former area to the latter gives μ.

Langley[2] and others have employed graphical methods for obtaining ratios of radiation weighted in different ways.

[1] Ives, Phil. Mag., Dec., 1912.

[2] Sci., June 1, 1883; Phil. Mag., May 30, 260, 1890.

Second. A screen with an aperture of suitable form may be used. This will be made clear by the following considerations:

Suppose we had a spectrum which had equal quantities of energy in the regions of every wave-length, and consider the radiation that passes through an aperture of the shape of the luminosity curve. The energy at each wave-length will be proportional to its light-giving power.

If, now, the spectrum of any source in question is formed with the same dispersing apparatus and allowed to pass through a screen with this aperture, the radiation from the source will be properly rated. We measure the energy in the spectrum after passing through the screen, and also the total energy without the screen. The ratio of the former to the latter is the radiant luminous efficiency sought. In unpublished work this method was suggested by Dr. Pfund in 1906; and also by Strache[1] in 1911.

Third. An absorbing solution may be made whose transmission curve is identical with the luminosity curve. Houstoun[2] has suggested the method. This is the method fully developed and described in this paper.[3]

Fourth.[4] Ives[5] has suggested the possibility of finding a photoelectric cell properly screened whose sensibility for various wave-lengths is of appropriate relative magnitude. Coblentz[3] suggests the possibility of finding a photoelectric radiometer with a sensibility curve that of the eye.

Fifth. A calorimetric method. This is described by Nichols[4] and it is not essential to describe it here. The purpose of the present investigation was to construct a cell with suitable absorbing solutions to be used in the direct determination of luminous efficiency. The transmission of such a cell must be such as to allow quantities of the radiation of any wave-length to pass through proportional to the luminosity of that wave-length; that is, the transmission curve must be identical with the luminosity curve of the average eye as given by Ives. The thermocouple screened with this cell in series with the galvanometer will respond to the radiation proportional to the true light value, that is, as the eye responds to that radiation giving the sensation of light.

The investigation is divided into two parts:

Part 1. An investigation of the transmission solutions contained in a quartz cell. The general method is obviously to measure the incident

[1] Proc. Amer. Gas Inst., 2, 401, 1911.

[2] Proc. Roy. Soc., 85 (A), 275, 1911.

[3] Phil. Mag., p. 853, Dec., 1912.

[5] Nichols, Lab. Manual, Vol. II., p. 325.

[3] A photographic method might be suggested here, with proper plates or plate in conjunction with suitable screen.

[4] Bul. B. of S., Vol. 9, page 46, 1913.

and transmitted radiation; and for this purpose a thermocouple was used. The scheme of apparatus used in this part is shown in Fig. 1.

The current through the Nernst glower was maintained at 0.8 ampere by potential from a storage battery. The carbon arc was tried as a source for radiation in the violet, but was found entirely unsatisfactory due to unsteadiness. The Nernst glower was used throughout the investigation of the transmission of solutions. The glower was protected from air currents by a small wooden box lined with heavy asbestos, which was just sufficiently large to receive the metal casing in which the filament

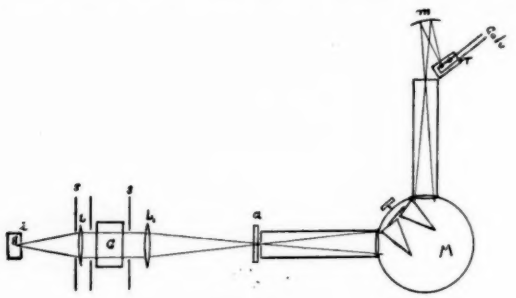

Fig. 1.

g = Nernst glower in inclosure i.
l, l_1, lenses.
s, s_1, screens.
c, cell containing solutions to be investigated.
a, auxiliary cell.
M, monochromatic illuminator.
m, silver concave mirror.
T, thermo-element.

was mounted. The box has a brass door with an aperture to allow full illumination of the first lens surface.

The lenses l, l_1 had a focal length of 14.5 cm. and diameter of 4 cm. Screen s was of heavy cardboard with a circular aperture of diameter somewhat less than 4 cm. so that the side walls of the cell (c) were not illuminated.

The cell (a) contained clear distilled water or a concentrated solution of copper chloride; the former when the longer wave-lengths were used, the latter when short wave-lengths were used. This auxiliary screen was placed as near as possible to the receiving slit of the illumination. It was designed to avoid the vitiating effects of scattered radiation.

The monochromatic illuminator was calibrated by means of the following lines: potassium 7,682, lithium 6,708, sodium 5,893, mercury (arc) 5,780, 5,461, 4,916, 4,358, 4,078. This calibration was checked several times during the work to detect any instrumental shifts. The light from

the transmitting slit was focused upon the thermocouple (T) by means of a short-focus concave silvered mirror (m). This was kept well polished always.

The thermocouple used in this part of the work was one constructed by Dr. Pfund[1] and used by him in measuring stellar radiation.[2] The thermo-element was enclosed in a sealed tube with a fluorite window. Evacuation was accomplished by means of Pfund's charcoal evacuator.[3] The degree of exhaustion was tested several times during the progress of the work and no diminution was noted even after months of service. For a detailed description the reader is referred to the original papers.[3]

The thermo-element was fastened securely in a box. Short leads connected its terminals with two large copper plates, also securely fastened in the box. Two additional wires led from the copper plates backwards. The box was then entirely filled with fine sand, to protect the thermo-element from convection currents. The copper plates were designed to act as heat reservoirs, so that conduction of heat energy to the thermal-element through the external leads would have little effect upon the galvanometer system. Over this box—with a suitable aperture to receive the light—was placed another box lined with heavy paper.

The galvanometer used in conjunction with the thermo-element was of a D'Arsonval type and was constructed by Dr. Pfund. It was designed to be quite stable and have sensibility sufficient for the present purpose. The suspension was of silver and about 10 cm. long. It was provided with a concave mirror approximately 2.5×3.5 mm. and of long focal length. Its period on open circuit was 15 sec. The galvanometer was enclosed in a box mounted upon an outside brick wall. The box was loosely packed with cotton. The mirror was seen through an aperture in the front of the box. This was covered with a thin strip of glass during the greater part of the work. The scale mounted upon ground-glass was four meters from the galvanometer and was supported upon a bracket securely fixed to a brick wall also. The image of a filament of an incandescent lamp was formed upon the ground-glass. The deflections were read with a hand glass magnifying about two and one half times. Later in the work advantage was taken of the diffraction images formed by the small mirror. With the reading lens the maxima and minima were very sharp and very easily seen.[4] The first dark space was about

[1] Phys. Zeit., 13, 870, 1912; PHYS. REV., XXXIV., 228, 1912.

[2] Pub. Allegheny Observatory, III., No. 6, p. 43.

[3] Phys. Zeit., 13, 870, 1912.

[4] Many maxima and minima could be seen sharply defined on either side of the central bright image.

the width of the lines on the scale. This first dark space was used instead of the bright central image. This proved to be a very convenient and accurate method of reading the deflections. The method was first suggested and tried by Dr. Pfund. A sliding holder for the lens was constructed so that the diffraction image could be followed more comfortably as the mirror deflected. A little scale rider graduated to tenths of millimeters was made. This could be pushed along with the lens to aid in estimating fractions of a millimeter. For the most part tenths of millimeters were estimated without the aid of the rider. The galvanometer system was remarkably free from accidental disturbances. There was very little drift of the zero point except owing to the extreme variations in the room temperature from morning to night. During the latter part of the work the space between the scale and the galvanometer was enclosed by means of tin pipe to avoid the scintillation of the image due to convection currents in the air. The galvanometer with the thermo-element in series with it was just critically damped. This condition was accurately brought about by adjusting an iron bar across the poles of the field magnet. The sensibility of the galvanometer and thermo-couple was such as to give 26 cm. deflection when a candle was placed at a distance of 1 meter in front of the thermo-element.

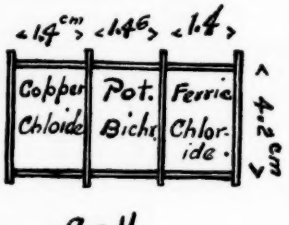

Fig. 2.

The deflections of the galvanometer were accurately proportional to the energy falling upon the thermocouple. This had been established by Dr. Pfund in a long series of tests made in the course of previous work.

The cell which contained the solutions investigated had three compartments as shown in Fig. 2. It was made from three rings cut from ink bottles. The rings were ground to have approximately parallel edges. They were cemented, by means of Canada balsam, end-on with quartz plates between and at both ends. The plates between the rings were 42 × 42 × 1 mm. while those at the ends were circular 42 mm. in diameter and 2 mm. thick. An opening was cut into each ring for purposes of filling. The quartz plates were purchased from The John A. Brashear Co. and the Bausch & Lomb Optical Co. They were only approximately plane parallel, but they were so placed on the rings that their individual defects were largely neutralized. The quartz plates were adjusted also so that the completed cell had quite perfectly parallel surfaces. This was tested by inserting the cell filled with water in front of a telescope focused on a distant church steeple. There was some

prismatic effect in the vertical plane; but this was not objectionable since it would cause a displacement of the image of the Nernst filament lengthwise along the slit of the monochromatic illuminator. When only one or two solutions were investigated the other compartment was filled with water. All the data given below were obtained with this cell. Other cells had been constructed and tried. They were made of plane parallel glass, but slight selective absorption of wave-lengths between .56 μ and .59 μ was noticed. On this account the cells were discarded. Quartz, on the other hand, offers a great advantage in that it can be readily duplicated.

The method of making determinations was as follows:

The solutions were put into the cell; the graduated head of the illuminator set to give the desired wave-length; the thermoelement was exposed to this wave-length first when the above cell was in the optical path and, second, when it was out; the ratio of the deflection in the first case to the deflection in the second case gives the percentage of transmission.

Determinations of this kind were made throughout the visible spectrum at intervals of .01 μ. These percentages of absorption were plotted against wave-lengths, that at .55 μ being taken as unity because this is the wave-length of maximum luminosity. This curve could then be compared with the luminosity curve[1] plotted to the same scale. To obtain a rough idea of the transmission curve of any set of solutions readings were made at .55 μ, .51 μ, .48 μ, .60 μ and .64 μ. This was sufficient to show what must be done to the solutions to approximate the luminosity curve, or whether they were suitable at all.

The slits used were as small as possible and still allow deflections accurately readable. In the extreme blue the widths were approximately 0.4 to 0.6 mm.; in the longer wave-lengths much less—in the red less than 0.1 mm.; at wave-length of maximum luminosity the slit width was approximately 0.2 mm. No corrections were applied for slit width.

There was always present a slight amount of stray light, through the monochromatic illuminator. In the final cell, however, with its high transmission, this was comparatively small. In the extreme blue, where large slits had to be used, another method was adopted as a check. A thin glass cell was made, filled with concentrated potassium bichromate solution. This was inserted immediately in front of the first slit of the illuminator. The opaque screen was raised; and, when the galvanometer became steady, the little cell was suddenly removed. The additional deflection was noted. This was entirely due to the blue at which the

[1] H. E. Ives, "Spectral Luminosity Curve of the Average Eye," Phil. Mag., Dec., 1912.

illuminator was set and which was out when the bichromate screen was in place. To avoid any displacement of the image of the Nernst filament and change in focus when the bichromate screen was taken out and replaced, the latter was immersed in a water cell which was always in front of the slit. The optical path was thus kept practically constant.

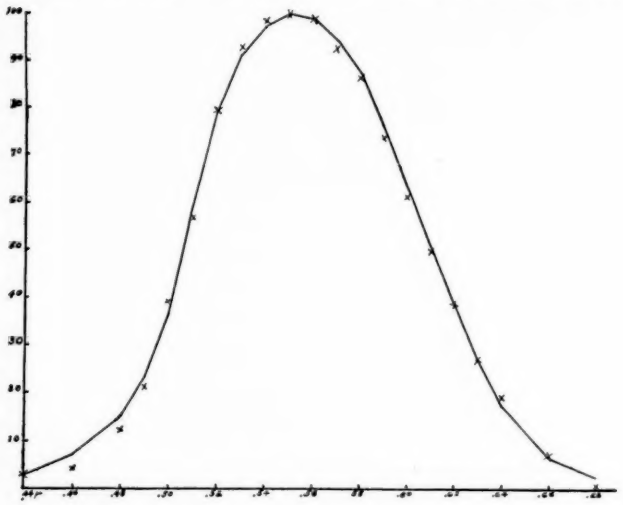

Fig. 3.

Transmission curve of cell in crosses Luminosity curve in full line.

It was found that the shifting of the zero noticed at the beginning of a series of observations could be eliminated quite completely by opening the slits and exposing the thermocouple for several minutes to the radiation from the Nernst glower. After this exposure the zero would remain constant.

Many inorganic salts were investigated; but only a few organic ones. Over two dozen dyes were examined in various combinations. The common difficulties with the dyes are that their absorption is high; they have sharp bands; and usually those having the adequate absorption near .60–.64 μ become again transparent in the infra-red. An additional objection is their instability and the uncertainty of exactly reproducing them.

The solutions that were found to answer requirements most satisfactorily with the cell above described were:

1. Cupric chloride ($CuCl_2 + 2H_2O$) purchased from Eimer & Amend. Analysis showing some trace of iron (0.0025 per cent.) and of sulphur (0.0016 per cent.) 41.085 grams of the salt were dissolved in one litre of distilled water. This solution was contained in the first compartment of cell 1.4 cm. in thickness.

2. Potassium bichromate ($K_2Cr_2O_7$, M. W. 294.5) purchased from J. T. Baker & Co., N. J. Analysis showing traces of calcium (0.001 per cent.) and of chloride (0.0001 per cent.) and sulphite (0.01 per cent.) 0.83462 gram per liter. This was contained in the middle compartment of the cell 1.46 cm. thick.

3. Ferric chloride ($Fe\ Cl_3$, $6H_2O$; M. W. 270.32) purchased from J. T. Baker & Co., N. J. Showing by analsyis P—trace, SO_3 0.002 per cent., HNO_3—trace, As—trace. 5.8712 grams per liter. Contained in the third compartment of the cell 1.4 cm. thick.

The concentrations of these solutions given above for thickness actually used become for a thickness of 1 cm. as follows:

Cupric chloride	57.5190 gm./l.
Potassium bichromate	1.2190 "
Ferric chloride	8.2200 "

The values for the transmission could be duplicated to within one per cent. on the red side of the maximum and to within four or five per cent. on the blue side. After four or five days the ferric chloride solution becomes opaque. Throughout the measurements made here this solution was prepared anew every other day. A certain percentage solution of iodine and potassium iodide can be substituted for the ferric chloride solution. This solution is .0515 gm. iodine with .4047 gm. potassium iodide to one liter of water. This iodine solution was used several days after preparation. It was contained however in a sealed jar. Whether there is any preponderating advantage to substitute it for the iron solution used cannot be stated at this time.

The values given in Table I. for the transmission are for the most part the means of four sets of data taken when the cell had been filled anew with new solutions. In each set three or four readings were taken at each wave-length. The measurements were taken at room temperature (about 19° C.). The temperature coefficient is too small to cause any noticeable change in absorption for a variation of a few degrees. The variations from the mean are less than 5 per cent. with the exception of three readings at 52, 54 and 55; considerably less in most cases, particularly on the red side of the maximum.

The deflections obtained without the cell were: At .50 μ 27.7 mm.; at .55 μ 39 mm.; at .60 μ 29 mm.; at .64 μ 45 mm.

Table I. shows the absolute transmission T in per cent. and the relative transmission T_r. Under V are given the corresponding values of the luminosity.[1] In column 1 are the wave-lengths; in column 2, the percentage transmission; in column 3, relative transmission, with that at .55 μ taken as 100; in column 4, the corresponding values for the

[1] Ives, Phil. Mag., Dec., 1912, p. 859.

TABLE I.

λ	T	T_r	V
.44μ		2.0	2.9
.46		4.0	7.3
.48	7.4	12.3	15.4
.49	12.8	21.3	23.5
.50	23.6	39.2	36.3
.51	34.2	57.0	59.6
.52	47.6	79.3	79.4
.53	55.9	93.0	91.2
.54	59.1	98.5	97.7
.55	60.0	100.0	100.0
.56	59.6	99.2	99.0
.57	55.4	92.5	94.8
.58	51.9	86.5	87.5
.59	44.4	73.8	76.3
.60	37.1	61.5	63.5
.61	30.0	50.0	50.9
.62	33.3	38.7	38.7
.63	16.3	27.1	27.2
.64	11.7	19.5	17.5
.65	4.3	7.1	6.8
.68	1.0	1.7	2.6

luminosity taken from Ives.[1] Below .48 μ and above .68 μ the curve is extended, for no readings here were taken. At .70 μ there was practically zero transmission; beyond .70 μ there was also zero transmission. The amount of energy given by the Nernst glower in wave-lengths below .48 μ is so small that a very large slit must be used and the measurements are very inaccurate. The transmission, however, extends down to .43 μ as is shown by the fact that the line of the mercury arc at .43 μ can be plainly see through the cell.

In Fig. 3 are given the transmission curve in crosses, and the luminosity curve of the average eye as found by Ives in full line. By means of a planimeter the areas of these curves were obtained. The area of the transmission curve is about 2.9 per cent. smaller than that of the luminosity one. It will be observed, however, that it is in the extreme blue where the maximum difference between the curves exists. In this region most light sources are deficient, so that the errors committed by using the cell is certainly much less than this.

The value of a knowledge of the radiant luminous efficiency of a light source may not be obvious to the reader who is not engaged in photometry, or who is not a worker in the profession of the lighting engineer. This quantity gives us a very interesting and valuable characteristic of

[1] Ives, Phil. Mag., Dec., 1912, p. 859.

the light source. It has close associations obviously with the total luminous efficiency, *i. e.*, the ratio of the radiations rated according to their luminosity to the total energy-input into the light source. For, knowing the losses (*e. g.*, in the case of the filament lamp, losses due to convection, conduction and resistance of the lead-in wires) the total luminous efficiency may be found from the radiant luminous efficiency.

Knowing the radiant luminous efficiency we would also know the total luminous efficiency from another consideration. For if we knew the watt equivalent of unit flux density of light of any wave-length we can also readily find the total luminous efficiency. The quantity to which reference has just been made has been termed the mechanical equivalent[1] of light. In our calculations we must take the "mechanical equivalent" of the radiation at wave-length of maximum luminosity (550 $\mu\mu$).

Calling the lumens per watt or radiation of wave-length 550 $\mu\mu$, K_{max}, we have

$$\frac{1}{K_{max}} = \text{watts per lumen}$$

or

$$\frac{4\pi}{K_{max}} = \text{watts per mean spherical candle.}$$

The radiant luminous efficiency (μ_R) tells us that only μ_R per cent. of the total radiation is light when the radiation is properly valued. Therefore,

$$\frac{4\pi}{K_{max}} \div \mu_R = \text{watts per mean spherical candle.}$$

Again, to fully and definitely describe a light source this quantity must be known. This will be seen by the following considerations. We may have two light sources which have the same value for the total luminous efficiency, *i. e.*, they may have the same energy-input and give the same amount of light radiation. Yet they may have quite different values of radiant luminous efficiency. The total luminous efficiency gives us the quantity which is most important from the economic standpoint; but is not by any means the only characteristic which we wish to know and must know for further development of our light sources.

In order that this quantity may mean what it should, however, all effects of radiation from mechanical parts connected with the source must be eliminated. The energy radiated only from the primary incandescent or luminous parts must be considered. This means that correc-

[1] Since the above work has been done Dr. Ives, Dr. Coblentz and Mr. Kingsbury have made an accurate determination of this quantity.

tion must be made for radiation from the chimney, from the bulb or other enclosure, and from the envelope of hot gases in case of certain sources. For this reason great accuracy for the values given below cannot be claimed. Correction was made only for one of the sources studied. The full significance of what errors due to this cause may be was fully realized in case of the mercury arc. Possibly to the illuminating engineer it is of greater usefulness to know the radiant luminous efficiency of the light source as it is used, that is without the corrections above applied.

It is obvious that the radiation measurements are not measurements involving in actual procedure the whole of the energy of radiation from the source; a particular direction and a particular solid angle are chosen. The direction is such as is usually of greatest interest for other measurements of the light sources studied.

Strictly in accordance with definition the whole energy of radiation in each case should be considered. The values obtained for the radiant luminous efficiency may be quite different in different directions. This must be so in case of the carbon arc and mercury arc and acetylene burner.[1] This has not been investigated and the author is not aware that any work or statement has been made upon this point. We might define other quantities such as the mean horizontal and the mean spherical value of the radiant luminous efficiency to be in correspondence with the measurements of candle power.

Part II. With this cell it was purposed to determine the radiant luminous efficiency of various light sources. This has only in part been

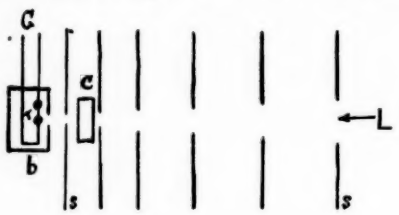

Fig. 4.

T, thermo-element with enclosure *b*.
s . . . *s*, screens.
g, galvanometer.
l, light source investigated.
C, cell.

accomplished; a few of the many sources have been studied, and all necessary corrections perhaps have not been applied.

The general arrangement of the apparatus for this portion of the work is shown in Fig. 4 above.

[1] Bul. Bur. Stand., Vol. 7, 1911.

This portion of the work was carried out, in most part, in the research laboratory of the United Gas Improvement Co. of Philadelphia, where access was had to various standard sources authoritatively rated. The new type of Leeds & Northrup galvanometer was used. The scale and telescope were at a distance of two meters from the galvanometer. This was made possible by placing immediately in front of the galvanometer a spherical concave spectacle lens (.25). Tenths of millimeters could easily be estimated for the image of the scale was almost natural size.

A Ruben's thermopile was used. This was employed instead of Pfund's thermocouple to avoid the uncertainty of absorption by the fluorite window for the various light sources.

About 22 cm. deflections were obtained with a candle at the distance of a meter. There was a very great lag, however, and a steady drift of the zero. Though the critical damping resistance of the galvanometer was 32 ohms this damping was not present when much greater resistance was inserted. To overcome this and the uncertainty as to what the effective distance was with a metallic cone before the junctions, the thermopile was dismantled of all its metal parts and mounted on a block of wood enclosed in a small pasteboard box with suitable aperture. Fine short leads connected it to two large pieces of brass cylinder; from which in turn two fine leads extended outwards. This box was loosely packed with cotton and enclosed in a second larger box mounted upon a long photometer bench constructed for the purpose. This last was also packed with cotton. There was now much less lagging and drift, though a considerable loss of sensibility was sacrificed. A series of screens was made of heavy cardboard mounted upon suitable blocks to fit the photometer bars. These screens had apertures of gradually decreasing size. They were stationed at intervals of from 10 to 20 cm. along the bar between the light source and the thermocouple.

The cell was mounted upon a small frame for which an appropriate sliding space with stops was arranged, so that the cell would be pushed aside and replaced in exactly the same position. The cell was placed as near as possible to the thermopile. Accurately round apertures in a thin board were painted black and placed immediately before and behind the cell. These apertures were a trifle smaller in diameter than the cell, to prevent any reflected light not going through the cell from reaching the thermopile.

The angle subtended by the thermopile at a distance of 63 cm. was one whose cross-section was 4 cm. horizontally by 22 cm. vertically. This angle was sufficiently large to include the whole surface of most of the illuminants investigated

The galvanometer deflections in the ranges occurring throughout the experiment were proportional to the energy falling upon the thermocouple. This was borne out for some very large deflections where the deviation was well under one per cent. The time required to reach a maximum deflection was approximately twenty-five seconds.

To take a set of measurements the light source was moved to a convenient distance; the thermoelement was exposed by raising a screen of heavy card suitably connected to a system of pulleys easily operated by the observer at the telescope; the maximum deflection was noted (the zero before and after being also noted); the cell was now pushed into position; the light source was moved sufficiently nearer, and a similar set of readings taken; the cell was then removed; the light source moved out to its first position and readings repeated. From ten to twenty readings were taken for each setting. For most of the sources recorded a similar series was taken on the two different days. There

TABLE II.

Welsbach Mantle Ceria 2 Per Cent. with Cell.

Zero.		D	r_0
Before.	After.		
2.30	2.10	3.70	1.50
2.10	2.10	3.65	1.55
2.10	2.10	3.80	1.70
2.10	2.13	3.75	1.64
2.13	2.25	3.86	1.67
2.25	2.35	3.97	1.67
2.35	2.50	4.11	1.69
2.65	2.80	4.45	1.73
2.80	2.90	4.60	1.75
2.90	3.05	4.55	1.58
2.42	2.60	4.16	1.65
2.60	2.62	4.25	1.64
2.62	2.70	4.32	1.66
2.70	2.78	4.15	1.41
2.78	2.90	4.55	1.71
2.90	3.00	4.56	1.61
3.00	3.05	4.70	1.68
3.05	3.20	4.75	1.63
			Mean 1.63

was always some drifting of the zero; and, though the galvanometer was mounted in the basement on the stone wall, accidental deflections due to mechanical disturbances from the mechanic's shop on the first floor were often considerable. The galvanometer suspension was very short,

which made such effects much more pronounced. The errors resulting from these sources were reduced by the large number of observations made.

Results.—A series of typical readings for the Welsbach mantle (2 per cent. ceria) with the cell is given in Table II.; for the same without the cell in Table III.; for the carbon point source ($4w/c$) readings with and

TABLE III.

Mantle 2 Per Cent. Ceria. Without Cell.

Zero.		D	r_1
Before.	After.		
2.15	2.10	24.46	22.34
2.10	2.32	24.82	22.61
2.32	2.40	24.95	22.59
2.40	2.25	24.96	22.62
2.25	2.08	24 90	22.74
2.08	1.85	24.75	22.79
1.85	1.82	24.60	22.77
1.82	1.70	24.49	22.73
1.70	1.58	24.47	22.83
1.58	1.40	24.30	22.81
1.40	1.38	24.20	22.81
1.10	0.93	23.70	22.69
0.93	0.80	23.75	22.92
0.80	0.43	23.55	22.91
2.35	1.95	25.50	23.35
1.80	1.55	24.47	22.80
1.55	1.30	24.40	22.98
1.20	0.95	24.12	23.05
0.95	1.10	23.98	22.96
1.10	0.95	24.05	23.03
0.67	1.10	24.45	23.57
1.00	1.00	24.25	23.25
0.85	0.70	24.15	23.43
0.40	0.20	23.80	23.50

without the cell are given in Table IV., and Table V. respectively; for the new tungsten-nitrogen filled lamp (.65 w/c) in Tables VI. and VII. respectively.

In columns one and two of each table are given the positions of the zero point before and after deflection; in column three, the observed maximum deflection; in column four, the true maximum deflections (r); (*i. e.*, column three corrected for position of "zero" as given by columns one and two).

In Table VIII. the means for all the sources studied are given, together with the corresponding distances and the ratio μ defined as the radiant

TABLE IV.

Carbon 99 v. (4 w/c). With Cell.

Zero.		D	r_v
Before.	After.		
2.47	2.50	4.75	2.27
2.50	2.48	4.78	2.29
2.48	2.48	4.70	2.22
2.48	2.57	4.80	2.28
2.57	2.55	4.81	2.25
2.55	2.58	4.82	2.26
2.58	2.64	4.90	2.29
2.64	2.70	4.99	2.32
2.70	2.72	5.00	2.29
2.72	2.80	5.01	2.25
2.80	2.87	5.17	2.34
2.87	3.00	5.24	2.31
3.00	3.05	5.37	2.35
3.00	3.08	5.33	2.29
3.08	3.13	5.40	2.30
3.13	3.15	5.43	2.29
3.15	3.20	5.48	2.31
3.20	3.30	5.58	2.33
3.30	3.40	5.63	2.28
3.40	3.40	5.70	2.30
			Mean 2.28

luminous efficiency. In column one are given the deflections (r_v) with the cell in place and with the source at a distance (d_v) as given in column three; corresponding quantities r_t and d_t without the cell are given in columns two and four. In column five is given the efficiency for each source.

The distances d_v are corrected for the change in optical distance due to the glass and water of the cell. This amounts to 0.7 mm. and was subtracted from the distances measured. All distances and deflections are given in centimeters.

μ has been defined above as

$$\frac{\text{Radiated energy} \times \text{luminosity}}{\text{total radiated energy}}.$$

With a cell whose transmission is identical with the luminosity curve this becomes

$$\frac{\text{Radiated energy} \times \text{transmission}}{\text{total radiated energy}}.$$

$$\mu = \frac{\int E_\lambda V_\lambda d\lambda}{\int E_\lambda d\lambda} = \frac{\int E_\lambda A_\lambda d\lambda}{\int E_\lambda d\lambda},$$

where A_λ is the absorption by the cell at wave-length λ.

TABLE V.

Carbon 4 w/c 99 v. Without Cell.

Zero.		D	r_t
Before.	After.		
0.00	−0.12	50.30	50.36
−0.12	−0.28	50.36	50.56
−0.28	−0.62	50.33	50.78
−0.62	−0.60	50.20	50.81
−0.60	−0.60	50.50	51.10
−0.60	−0.30	50.48	59.93
−0.30	−0.60	50.32	50.77
−0.60	−0.65	50.12	50.74
−0.65	−0.80	50.30	51.02
−0.80	−0.50	50.21	50.86
−0.50	−0.88	50.28	50.97
−0.88	−0.78	50.45	51.25
−0.78	−0.75	50.10	50.86
−0.75	−0.53	50.22	50.86
−0.53	−0.60	50.25	50.82
−0.60	−0.62	50.50	51.11
−0.67	−0.62	50.40	51.04
−0.62	−1.20	50.13	51.04
−1.20	−0.72	50.20	51.16
−0.72	−0.70	50.12	50.83
−0.70	−1.40	50.80	51.85
			Mean 50.93

In Table VIII. above r_v is a measure of the energy after going through the cell and r_t is a measure of the total radiation, but to make these comparable they must be corrected. First, for distance. The deflections r_v and r_t must be reduced to the values they would have had if the source had been at some fixed distance for both (unit distance, say). That is, r_v must be multiplied by d_v^2 and r_t by d_t^2. Second, with the cell in, all of the radiation at any wave-length does not have effect upon the thermo-couple for a certain amount is absorbed by the cell. Experiments made with radiation of wave-length .55 μ proved that only 60 per cent. of the energy at that wave-length is transmitted; hence, r_v must be multiplied by 100/60. Therefore

$$\mu = \frac{100\,\frac{r_v \times d_v^2}{60}}{r_t \times d_t^2}.$$

The values of μ given in column five are calculated from this equation for each source.

The radiation from the whole lamp was in each case allowed to fall

TABLE VI.

Tungsten (.65 w/c) 6.6 amp. With Cell.

Zero.		D	r_0
Before.	After.		
7.02	7.10	14.02	6.96
7.10	7.20	14.00	6.85
7.20	7.28	14.12	6.88
7.28	7.26	14.20	6.92
7.26	7.40	14.20	6.87
7.40	7.40	14.30	6.90
7.40	7.45	14.34	6.92
7.45	7.40	14.30	6.88
7.40	7.49	14.28	6.84
7.49	7.60	14.57	7.03
7.60	7.63	14.53	6.92
7.63	7.75	14.63	6.94
7.68	7.75	14.55	6.84
7.75	7.80	14.75	6.98
7.80	7.85	14.52	6.70
7.85	7.80	14.70	6.88
7.80	7.85	14.70	6.88
7.85	7.90	14.76	6.89
7.90	7.89	14.70	6.81
7.89	7.88	14.90	7.02
7.88	8.00	14.85	6.91
8.00	7.95	14.95	6.98

upon the thermopile. No correction was made for the radiation from any portion of the mechanism, which is an essential part of the light source as ordinarily used. For example, no correction was made for the radiation from the chimney of the Welsbach mantle lamp. For an accurate determination of the efficiency of the mantle itself such a correction must be made, by getting a cooling curve of the chimney.[1] Such a correction would increase the efficiencies recorded above. The incandescent electric lamps used were matched in color on the photometer by Mr. Kingsbury of the photometrical laboratory of the U. G. I. Co. against standards of approved type prepared by the Electrical Testing Laboratories, New York City.

The values for the tungsten lamps are inversely proportional to the watts per candle ratings. This is to be expected since the losses in incandescent lamps are proportional very approximately to the watts per candle.

A similar calculation based upon watts per candle for the new tungsten and for the carbon filament lamps, taking the efficiency of the tungsten

[1] Nichols, Laboratory Manual, Vol. II., p. 335.

TABLE VII.

Mazda (.65 w/c) 6.6 amp. Without Cell.

Zero.		D	r_t
Before.	After.		
9.20	8.98	35.70	26.62
8.97	9.75	36.25	26.89
9.75	9.25	35.80	26.30
9.25	8.60	35.80	26.38
8.60	8.75	35.15	26.48
8.75	8.65	35.65	26.95
8.65	8.55	35.40	26.80
8.55	9.30	35.75	26.33
9.30	8.60	35.90	26.95
8.60	8.65	35.15	26.53
8.65	9.22	35.60	26.67
9.22	9.20	36.12	26.91
9.20	9.00	35.20	26.10
8.88	8.95	35.45	26.54
8.95	10.00	36.20	26.73
10.00	9.45	36.45	26.73
9.53	8.85	36.05	26.86
8.85	7.80	35.05	26.73
7.80	8.15	34.96	26.99
8.15	7.45	34.60	26.80
7.80	7.72	34.48	26.67

(1.25 *w/c*) as 1.65, does not give the actual values observed. For the new tungsten lamp the value is 3.25, showing that the losses must be considerably more, or that the determined value of μ is too low, due to the excessive heating of the bulb. No correction was made for the hot gaseous envelope surrounding the candle and the open burner. The latter was a lava-tip burner consuming about four cubic feet of gas per hour. It had to be brought very close to the cell to get suitable deflections. At this distance the whole of the flame could not contribute to the deflection. Correcting the value of μ given above for this, the value becomes approximately 0.3 per cent. The mantle with 0.25 per cent. ceria is commercially known as "Iveslite" mantle. The one with 0.75 per cent. ceria is the mantle commonly used.

The chimney referred to above as "perforated" chimney had air rents around its base, in distinction from the solid chimney which had none. The gas was city gas of Philadelphia known as mixed gas rated at 650 B. T. U. per cu. ft. approximately; pressure 2.5 inches of water.

The mantle marked 1)*a* in the table consumed 3.8 cu. ft. per hour; 1)*b* 4.95 cu. ft. per hr.; 2) 3.6 cu. ft. per hr.; 3)*a* 4 cu. ft. per hr.; 3)*b* 5.2 cu. ft. per hr.

TABLE VIII.

Results for Various Sources.

Source.	r_v	r_t	d_v	d_t	μ	Description.
Carbon filament.....	2.28	50.92	29.17	120.52	0.45	Point source 4 w/c 99v.
Nernst glower.....	4.65	28.46	20.39	102.90	1.08	0.8 amp. stereopticon type.
Tungsten-nitrogen.....	6.89	26.72	29.53	113.13	2.93	6.6 amp. (0.65 w/c)
Open burner.....	1.50	42.19	20.88	115.52	0.19	See below
Standard candle.....	0.25	8.67	16.45	73.85	0.24	Sperm candle
Mantles (1) *a*.....	0.89	21.34	44.8	163.3	0.5	0.25% Ceria
(1) *b*.....	1.48	23.57	44.9	163.4	0.7	0.25% Ceria
(2).....	1.62	22.91	43.77	162.9	0.8	2% CeO_2
(3) *a*.....	2.90	21.43	37.60	163.4	1.2	Solid Chimney
b.....	3.81	26.91	37.6	163.4	1.26	perforated, 0.75%CeO_2
Tungsten (1).....	3.65	16.28	27.75	119.47	1.99	102.6 v. 1 w/c
(2).....	3.11	14.99	27.75	119.47	1.84	97.0 v. 1.1 w/c
(3).....	2.51	13.53	27.75	119.47	1.65	9.16 v. 1.25 w/c
Mercury arc[1].....	0.44	2.4	30.0	30.0	30.5	1.7 amp.

From 1)*a* and 1)*b* mantles it is seen that the efficiency varies with the gas consumption. This is to be expected for when an insufficient supply of gas is admitted, certain portions of the mantle will be cool; also when too much gas is admitted a small bunsen flame may exist above the mantle; both these conditions will lower the value of μ. The values for the mantles with varying per cent. of ceria are what might be expected from a study of the curves showing per cent. of ceria with candle power.[2] The mantle when used first indicated a greater efficiency than after a few hours burning, but was much more unsteady. Just how the radiant luminous efficiency changes for small changes in consumption at the point where the mantle is just filled and at its brightest would require more accurate measurements and steadier conditions. Two other determinations were made for the 4 *w/c* carbon lamp which make the mean value of μ equal to .43 per cent.

The Heraeus quartz arc at 10 amperes was also tried for efficiency. The quartz tube was between 6 and 8 cm. long and a portion of the tube only used. A screen of heavy cardboard with an aperture 2 cm. horizontally by 3 cm. vertically was placed immediately in front of the arc, in such a manner that it could be dropped and put in place only when deflections were taken. This was to insure that the screen itself would remain at room temperature. Values obtained for the arc are unsatisfactory. Taking measurements immediately after starting gave a value

[1] Pfund's Type, PHYS. REV., N. S., Vol. III., No. 4, April, 1914.

[2] Johns Hopkins Lectures on Illuminating Engineering, M. C. Whittaker.

in μ in the neighborhood of 7 per cent. This value would steadily decrease for hours afterwards due to heating up of the tube.

Measurements were made in Baltimore with the new type of mercury arc recently described by Pfund.[1] The apparatus used was the same as that described in the first part of this paper. For Pfund's thermo-element as used, there was substituted the Ruben's thermopile as used in the measurements of the previous efficiencies. The procedure in this case was identical with the previous. The thermopile was carefully shielded and screens were placed at intervals between it and the source. The portion of the lamp just above the mass of mercury at the positive electrode was studied, the rest of the lamp being carefully screened off. This portion was 1.3 cm. wide by 1.8 cm. high. The lamp was placed 32 cm. from the thermopile and it was found unnecessary to move it. Even with the cell in place readable deflections were obtained. The current was 1.8 amp. taken from a storage battery with lamps in series as resistance.

Table IX. gives the zero position before and after in column 1; the position of maximum deflection, column 2; the magnitude of the deflections, column 3; both with the cell in place and without the cell.

To determine the correction to be applied due to the radiation from the walls of the tube, the lamp was put out and the time noted. Imme-

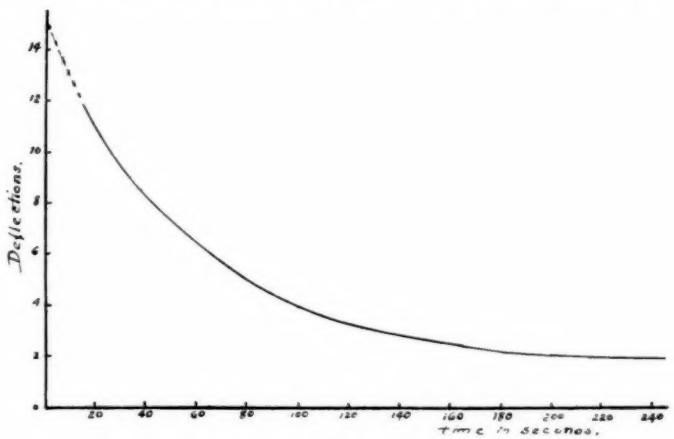

Fig. 5.

diately thereafter the screen was drawn aside allowing the radiation from the dark hot tube to fall upon the thermopile. The deflection and the time were again noted. Several other deflections were obtained and times at which they were gotten were noted, while the tube cooled. These deflections and times reckoned in seconds are plotted in Fig. 5.

By exterpolation the deflection due to the radiation from the walls just at the time the lamp was quenched was obtained. This deflection was subtracted from the deflections recorded for total radiation. From the corrected deflections the value of the efficiency was then calculated. This is rather surprisingly high. This may be partly accounted for by the fact that the energy in the extreme violet and ultra-violet is not effective under this arrangement. Without correcting for the hot walls the value is only 4.2 per cent. This type of mercury arc, however, affords a very nice means of getting the radiant luminous efficiency through the quartz plate sealed to the upper end of the tube. By using a right-angled quartz prism much of the effect due to the heated tube could be at once eliminated. No such measurement was made at this time.

The carbon arc was found too unsteady to get reliable measurements. Magnetite electrodes were received from Dr. Whiteney, of the General Electric Laboratory, and various luminous carbons from Mr. Chillis, of the National Carbon Co. It is hoped that these will be studied in the near future.

TABLE IX.

Zero With Cell, Mm.	P_m, Cm.	Deflection, Cm.
0, −.3	.40	.41
0, −.3	.40	.41
0, 0	.43	.43
0, −.5	.42	.44
0, −.5	.43	.45
0, —.3	.45	.46
0, — 0	.47	.47
0, −.2	.44	.45
0, — 0	.45	.45
		Mean.....44
Without cell		
0, 1.1	17.7	17.2
0, .4	17.4	17.2
0, .5	17.4	17.2
0, .5	17.9	17.7
0, .2	17.4	17.3
0, .2	17.5	17.4
0, .4	17.9	17.7
0, .2	17.4	17.3
0, .2	17.6	17.5
		Mean.....17.4

This work was suggested by Dr. H. E. Ives while the writer was working in the research laboratory of the United Gas Improvement Company of Philadelphia during the summer of 1913. To him the

writer is greatly indebted for suggestions, for the standard illuminants used and for the many facilities placed at his disposal while measurements on efficiencies were made.

The writer wishes to express his appreciation of the interest of Professor Ames in this work and of the other members of the physics faculty, especially of Dr. Pfund, to whom grateful acknowledgment is made for suggestions and encouragement throughout the work.

JOHNS HOPKINS UNIVERSITY,
June, 1914.

THEORY AND USE OF THE MOLECULAR GAUGE.

BY SAUL DUSHMAN.

SOME time ago Dr. I. Langmuir described the construction of a "molecular" gauge for the measurement of very small gas pressures.[1] At the suggestion of Dr. Langmuir the writer undertook a more detailed study of the theory and use of the instrument, and the following paper contains the results of a number of measurements that were carried out with the aid of this gauge.

THEORETICAL.

If a plane is moving in a given direction with velocity u relatively to another plane situated parallel to it at a distance d, there is exerted on the latter a dragging action whose magnitude may be calculated from considerations based on the kinetic theory of gases.

At comparatively higher pressures where the mean free path of the gas molecules is considerably smaller than the distance between the plates, the rate of transference of momentum across unit area is given by the equation

$$B = \frac{\eta u}{d}, \tag{1}$$

where η denotes the coefficient of viscosity.

According to the kinetic theory of gases this coefficient ought to be independent of the pressure. The confirmation of this deduction over a very large range of pressures has been looked upon as one of the most striking arguments for the validity of the assumptions on which the kinetic theory of gases is based.

It was found, however, by Kundt and Warburg,[2] that at very low pressures, where the mean free path of the molecules becomes of the same order of magnitude as the distance between a moving and stationary surface placed in the gas, there is distinct evidence of a slipping of gas molecules over the planes. The amount of this slip was found to be inversely proportional to the pressure.

[1] PHYSICAL REVIEW, *1*, 337 (1913). See also abstract, PHYS. REV., *2* (1913).

[2] Pogg. Ann., *155*, 340 (1875). Poynting and Thomson, Properties of Matter, p. 220.

Denoting the coefficient of slip by δ, it may be defined by the relation

$$v_g - v_s = \frac{\eta}{\zeta}\frac{\partial v}{\partial x} = \delta \cdot \frac{\partial v}{\partial x}, \tag{2}$$

where v_g = velocity of gas molecules at the surface,
v_s = velocity of surface,
ζ = coefficient of external viscosity.

It follows from hydrodynamical considerations that the amount of momentum transferred per unit area is

$$B = \frac{\eta u}{d + 2\delta}. \tag{3}$$

Thus, owing to slip there is an apparent increase in the thickness of the gas layer between the two surfaces. This increase amounts to $\delta = \eta/\zeta$ for each surface.

Experiments on the conduction of heat at low pressures led to similar observations in this case. According to the kinetic theory the heat conductivity should be independent of the pressure. Accurate determinations showed that at very low pressures the conductivity apparently decreases. This led to the conception that at the surface there occurs a very steep temperature gradient (Temperatursprung) so that the amount of heat, Q, conducted between two surfaces maintained at temperatures T_1 and T_2 is given by

$$Q = \frac{T_1 - T_2}{d + 2\gamma}, \tag{4}$$

where d is the distance between the two plates and γ represents the apparent increase in thickness of the layer of gas at each surface.

The definition of γ may be expressed by the following relation, which by its analogy with equation (2) helps to exhibit the complete parallelism of the phenomena observed in both the case of heat conduction and that of viscosity effect. Denoting the drop in temperature at the surface by ΔT, and the temperature gradient there by $\partial T/\partial x$, the definition of γ follows from the relation

$$\Delta T = \gamma \cdot \frac{\partial T}{\partial x}. \tag{5}$$

An interpretation of this temperature drop on the basis of the kinetic theory of gases was first advanced by Maxwell and subsequently developed still further by Smoluchowski.[1]

It is assumed that of the molecules striking a heated surface only a fraction f is absorbed and then emitted with an average kinetic energy

[1] Ann. Phys., *35*, 983, where references to previous literature are given.

corresponding to that of the surface. The remainder $1 - f$ is reflected according to the laws of elastic collision. If T_1 denote the temperature of the molecules striking the surface, T_2, the temperature of the latter, and T_2^1 the temperature of the molecules leaving the surface, then:

$$T_2^1 - T_2 = (1 - f)(T_1 - T_2). \tag{6}$$

The constant f is known as the coefficient of equalization (Smoluchowski) or accommodation (Knudsen),[1] for it is evident that f is unity when the average temperature of the molecules leaving the heated surface corresponds to the temperature of the latter.

In consequence of this lack of complete equalization of temperatures, there is produced an apparent temperature drop at the surface, which is related to the coefficient of equalization by the following equation:

$$\gamma = \frac{2 - f}{f} \cdot \frac{15}{4\pi} L, \tag{7}$$

where L is the mean free path of the molecules at the given pressure.

The same method of interpretation was extended to the case of transference of momentum from one surface to another at very low gas pressures. Of the molecules striking a moving surface a portion β is "absorbed" and then "emitted" with velocities that range according to Maxwell's distribution law. The direction of emission is perfectly independent of the direction of incidence. On the other hand, the fraction $1 - \beta$ is "reflected" according to the laws of elastic collision.[2]

A number of investigators have concerned themselves with the mode of determination of these coefficients β and f, which may be designated as the coefficients of accommodation for viscosity and heat conduction respectively.

Knudsen,[3] who has carried out a large number of investigations on the behavior of gases at very low pressures, concludes that while the value of this coefficient is less than unity for heat conduction (and differs with the nature of the gas) it is equal to unity in all those cases where transference of momentum is concerned. He assumes, in other words, that all the molecules are emitted from a moving surface in directions which are absolutely independent of the original directions of incidence and that these molecules then obey Maxwell's law of distribution of velocities.

Timiriazeff[4] makes the assumption that the coefficient of accommodation has the same value, both for viscosity measurements and for the

[1] Ann. Phys., *34*, 593 (1910).

[2] A. Timiriazeff, Ann. Phys., *40*, 978 (1913).

[3] Ann. Physik, *28*, 75 (1908); *31*, 205 (1909); *33*, 1435 (1910); *34*, 593, 823 (1911); *35*, 389 (1911); *36*, 871 (1911).

[4] *Ann. Physik*, *37*, 233 (1912).

determination of heat conduction in gases at low pressures and deduces from this assumption the relation

$$\delta = \frac{8}{15}\gamma \tag{8}$$

where δ has the significance assigned to it in equation (3) above.

From equations (7) and (8) it follows that

$$\delta = \frac{2}{\pi} \cdot \frac{2-f}{f} \cdot L. \tag{9}$$

More generally, we can write

$$\delta = a \cdot L$$

where a is a constant whose exact value depends upon the particular assumptions made regarding the value of the accommodation coefficient.

At extremely low pressures, where L is large compared to d, equation (3) reduces to

$$B = \frac{\eta u}{2aL} \tag{10}$$

or, since

$$\frac{\eta}{L} = 0.31 p \sqrt{\frac{8M}{\pi RT}},$$

$$B = \frac{2 \times 0.31}{a} pu \sqrt{\frac{M}{2\pi RT}}. \tag{11}$$

Substituting for a the value deduced by Timiriazeff, see equation (9), it follows that

$$B = \frac{f}{2-f} \times 0.31 \pi u p \sqrt{\frac{M}{2\pi RT}}. \tag{12}$$

A relation of the same form as this may also be deduced by means of considerations similar to those used by Knudsen. This method of derivation has the advantage that it does not involve any extrapolation of equation (3), but starts from fundamentally different premises.

At very low pressures, the mass of gas striking unit area of a surface per unit time is equal to

$$\frac{1}{4}\rho\Omega = p\sqrt{\frac{M}{2\pi RT}}$$

where

ρ = density of gas,

Ω = average (arithmetical) velocity.

Assuming, as Knudsen does, that the coefficient of accommodation is

unity, it follows that the rate of transference of momentum per unit area from a surface moving with velocity u is

$$B = up\sqrt{\frac{M}{2\pi RT}}. \quad (13)$$

Equations (12) and (13) agree in the conclusion that at very low pressures B is proportional to $p\sqrt{M/RT}$.

According to Gaede,[1] Knudsen's assumption that the accommodation coefficient is equal to unity in the case of viscosity measurements at very low pressures is justified only at pressures below about 1.33 bars (.001 mm. of mercury). In a very recent paper, Baule has discussed the work of Smoluckowski, Knudsen, Timirazeff and others in detail[2] and by introducing some very plausible assumptions as to the actual mechanism by which a gas molecule exchanges energy with a molecule of the surface against which it strikes, he arrives at the relation

$$B = up\left(\frac{(1 - \alpha'v)}{(1 + \alpha'v)}\right)\sqrt{\frac{M}{2\pi RT}}, \quad (14)$$

where $\alpha'v$ is a function of the masses and diameters of the molecules of the gas and solid, and the distances between the molecules in the plane surface. It is thus evident that no two writers are agreed upon the manner in which the coefficient f is to be calculated.

We are, however, justified in concluding that there exists a relation between B and $p\sqrt{M/RT}$ of the general form

$$B = kup\sqrt{\frac{M}{RT}}, \quad (15)$$

where k is a *constant* whose value depends upon the nature of the gas and that of the surface with which it is in contact.

This is the fundamental relation upon which is based the construction of the "molecular gauge" described in the following section.

DESCRIPTION OF GAUGE.

The construction of the gauge is shown in Fig. 1. It consists of a glass bulb B in which are contained a rotating disc A and, suspended above it, another disc C. The disc A is made of thin aluminum and is attached to a steel or tungsten shaft mounted on jewel bearings and carrying a magnetic needle NS. Where the gauge is to be used for measuring the pressure of corrosive gases like chlorine, the shaft and disc may be made of platinum. The disc B is of very thin mica, about

[1] Ann. Physik, *41*, 289.

[2] B. Baule, Ann. d. Physik, *44*, 145 (1914).

.0025 cm. thick and 3 cm. in diameter. A small mirror, *M*, about 0.5 cm. square is attached to the mica disc by a framework of very thin aluminum. This framework carries a hook with square notch which fits into another hook similarly shaped, so that there is no tendency for one hook to turn on the other. The upper hook is attached to a quartz fiber, about 2×10^{-3} cm. diameter, and 15 cm. long.

"The lower disc can be rotated by means of a rotating magnetic field produced outside the bulb. This field is most conveniently obtained by

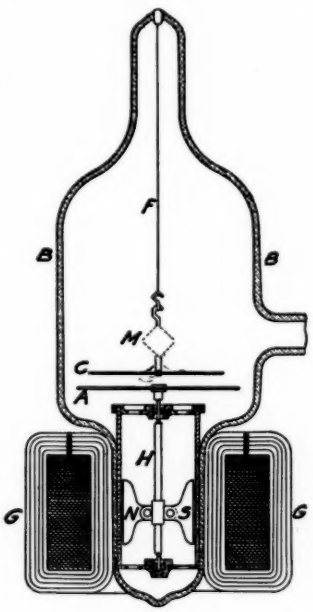

Fig. 1.

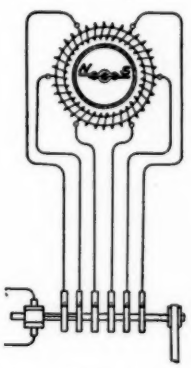

Fig. 2.

Rotating Commutator and Connections to Gramme Ring *G–G* of Fig. 1.

a Gramme ring (*GG*) supplied with current at six points from a commutating device run by a motor (see Fig. 2). In this way the speed of the motor determines absolutely the speed of the disc, since the two revolve in synchronism. The speed of the disc may thus be varied at will from a few revolutions per minute up to 10,000 or more."

In constructing the gauge, the lower part of the bulb is made to fit an aluminum spring holder which supports the spun aluminum cylinder. The latter contains the upper and lower jewel bearings on which the shaft (*H*) rotates. The bulb is then cut across the widest portion (at *BB*) and the two discs are introduced, care being taken to see that the framework which carries the mirror is not bent during the subsequent re-sealing of the two parts. The quartz fiber is threaded through the

hook and a little glass bead attached on the lower end, while the upper end is fastened to a platinum wire by means of sealing-in glass. The last operation consists in "fishing" for the mica disc by means of the hook on the quartz fiber, and after the distance between the two discs has been adjusted, so that the upper disc hangs centrally over the lower disc in a perfectly horizontal plane at a distance of less than 1 cm., the glass at the top of the bulb is closed up around the platinum wire.

One of the great advantages of the gauges as constructed in the above manner is *the complete absence of any parts that cannot be heated up to a temperature of about 300° C. No cement, shellac or other source of vapor should be used in attaching the mirror or the quartz fiber.*

CALIBRATION OF GAGE.

Let r = radius of rotating disc,

ω = angular velocity of rotating disc,

α = angle of torque of upper disc,

D = "Direktions-kraft" on upper disc

$$= \frac{\pi^2}{t^2} K,$$

where K = moment of inertia of upper disc,

t = period of oscillation.

From equation (15) it follows that the momentum transferred per unit time to upper disc is

$$B_0 = \int_0^r \omega r^2 \cdot 2\pi r dr \cdot kp \sqrt{\frac{M}{RT}}$$

$$= \frac{k\pi r^4 \omega p}{2} \cdot \sqrt{\frac{M}{RT}} \qquad (16)^1$$

$$= \alpha D = \frac{\alpha \pi^2}{t^2} \cdot K.$$

Consequently

$$\alpha = \left(\frac{kt^2 r^4}{2\pi_k}\right) pw \sqrt{\frac{M}{RT}}. \qquad (17)$$

Hence the torque on the upper disc is proportional to the product of the speed of rotation of the aluminum disc and the function $p\sqrt{M/RT}$.

Upon this equation depends the use of the instrument as a sensitive vacuum gauge.

[1] This equation is only rigorously true if the diameter of the rotating disc is very large compared with that of the upper disc, so that errors due to "edge effect" are avoided.

By properly designing the dimensions of the discs it is evident that equation (17) could be used for *very accurate determinations of the value of k.* In this manner the conclusions of Knudsen, Smoluckowski and Baule on the correction for slip could readily be tested. As the present investigation was carried out mainly with the view of determining the utility of the instrument as a gauge, no such accuracy was attempted so that definite conclusions could not be drawn regarding the value of k. In each case the gauge was calibrated at pressures of about .001 to .01 mm. of mercury against a McLeod gauge. The following data give, however, some idea of the degree of sensitiveness to be expected (and actually obtained) from a gauge constructed on the above principles.

For this particular gauge, the weight of the mica disc was 0.1 gm., $r = 2$ cm., $t = 12$ seconds.

Consequently

$$K = \tfrac{1}{2} W^2 = \tfrac{1}{2} \times 0.1 \times 4 = 0.2.$$

Assuming a speed of 1,000 r.p.m.,

$$\omega = \frac{2\pi}{60} \times 1{,}000.$$

In the case of air at a pressure of *1 bar*[1] and 300° Abs., it is found by making the proper substitutions in equation (17) and assuming $k = 1/\sqrt{2\pi}$ that

$$\alpha_{\text{calc.}} = 150° \text{ per bar.}$$

By illuminating the mirror and using a similar arrangement to that used for galvanometers, it is possible to detect a deflection of 1 mm. at a distance of 50 cm. or

$$\frac{1}{500} \times \frac{180}{\pi} \times \frac{1}{150} \text{ bar} = 0.8 \times 10^{-3} \text{ bar.}$$

Increasing the speed to 10,000 r.p.m. increases the sensitiveness tenfold and under these conditions it ought, therefore, to be possible to measure a pressure of about 10^{-4} bar.

CORRECTION FACTORS.

In using the instrument there are, however, several points regarding which special care ought to be taken.

1. *Correction Due to Eddy Currents in Metal Parts of Mica Disc.*—

[1] In accordance with most recent practice, we have adopted in this paper as unit of pressure 1 dyne per cm.^{2}. This is known as a *bar*. The relation between this unit and the conventional unit ($1\mu = 10^{-3}$ mm.) is very simple. For all purposes the relation $1\mu = 4/3$ bar is accurate enough. The exact relation is that 1 micron of mercury at 45° latitude and sea-level is equal to 1.01327/.76 = 1.33325 bar.

Owing to the rotation of the magnetic field produced by the Gramme ring, eddy currents are set up in the metal framework used to hold the mirror on the mica disc. Denoting the current through the commutator and Gramme ring by i, the torque actually produced on the upper disc may be expressed as additively composed of two terms, one due to the gas molecules from the rotating disc, and the other due to eddy currents in the metal parts of the upper disc. Consequently, equation (17) assumes the form

$$\alpha = \left(\frac{kl^2r^4}{2\pi K} p\omega \sqrt{\frac{M}{RT}} \right) + k_1 i^2 \omega, \tag{18}$$

where k_1 is a constant for the gauge.

The magnitude of the correction term may be diminished by using metal parts whose electrical resistance is very high and by placing the Gramme ring at a greater distance below the upper disc.

On the other hand, there is really no need for any metal parts whatever in connection with the upper disc. The mirror could be supported in a mica or glass holder, and where extreme accuracy is desired such a construction could no doubt be worked out in detail. For ordinary purposes where it is desired to measure pressures that are not less than 0.001 bar, a framework of thin aluminum wires for holding the mirror introduces no measurable errors.

2. *Synchronism.*—From the construction of the apparatus, it is evident that the aluminum disc rotates five times as fast as the magnetic field. In order to maintain the disc and commutator in synchronism, a rotating sector with five slots in it may be attached to the commutator so as to enable the operator to view a mark on the aluminum disc which should obviously appear to remain stationary if the two are in synchronism. An equally good check is to take readings of the deflection at different speeds. If the speed of the commutator is increased very slowly, there is no difficulty in maintaining the disc and commutator in synchronism.

4. *Relative Position of Discs.*—At a pressure of 1 bar and ordinary temperatures, the mean free path for air is about 10 cm. Consequently, in order that equation (17) should be valid at this pressure, the discs ought to be placed at a distance of less than 1 cm. apart. Care should also be taken to see that the upper disc is located centrally over the lower one. Regarding which disc should be the larger, the following considerations are of interest. In the operation of the gauge there is always a tendency for the upper disc to start swinging or at least get away from its symmetrical position with respect to the lower disc. If the latter is large compared to the mica disc, there obviously results a much greater torque on one side of the disc than on the other and the tendency to

swing is increased until finally the disc hits the walls of the bulb. As the damping at low pressures is very feeble, it is very difficult to stop this oscillation when once started, except by imparting to the bulb itself an opposing motion by hand. After a little experience it is easy in this manner to stop any tendency for the disc to vibrate.

Where it is intended to use the instrument as an absolute gauge or for the determination of k, it is obviously necessary to have the rotating disc much larger. On the other hand, for most purposes, that is where the instrument can be calibrated against say a McLeod gauge at pressures above 1 bar, and used to extrapolate the indications of the latter for very low pressures, it is more advantageous to have the upper disc larger since, in this manner, the tendency to swing is diminished considerably. It must be remembered, however, that as the area of the upper disc is increased beyond that of the lower disc the sensitiveness is decreased.

EXPERIMENTAL.

1. *Preliminary Experiments.*—The gauge used in these experiments contained a much heavier mica disc (weight about 0.5 gm.) and a phosphor-bronze suspension similar to those used in galvanometers. The deflection was determined directly by noting the position of a mark on the mica disc with respect to a circular scale outside the bulb. The molecular gauge was connected in series with a liquid air trap to a Gaede mercury pump and ordinary McLeod gauge.

The following data show that the deflections observed are proportional to the rate of rotation of the aluminum disc. Under p is given the pressure in bars; under r, the rate of rotation in r.p.m., and under A the deflection in degrees. The fourth column gives $D = (A/r) \times 1{,}000$, while the last cloumn gives $D_0 = (A/r) \times 1{,}000/p$, that is, the *deflection per bar at 1,000 r.p.m.*

TABLE I.

p	r	A	D	D_0
0.97	850	18	21	21.3
	1,200	28	23	23
	2,000	44	22	22.5
5.74	1,050	80	76	13.5
	1,750	120	70	12

The reason for the larger value of D at low pressures is probably due to the presence of water-vapor and other condensible gases in the gauge, as the bulb had not been previously baked out.

After allowing dry air to enter the system until the pressure was over

5 mm. of mercury, readings were taken of both the McLeod and molecular gauges, as the pressure was decreased by pumping.

The results of the observations are recorded in Table II.

TABLE II.

p	D	D_0
1,707	990	0.6
960	870	0.9
540	840	1.6
304	820	2.7
171	745	4.0
96	655	6.8
55	490	9
31	360	11.6
24.5	280	11.4
14.0	180	12.8
8	106	13.5
4.7	67	14.3
3.5	46	14.3

It will be observed that up to about 20 bar the deflection was proportional to the pressure. At this pressure the mean free path in air is about 0.5 cm., and this was about the distance between the two discs.

2. *Vapor Pressures of Mercury and Ice.*—For the observations recorded in this section, the sensitiveness of the molecular gauge used was such that $D_0 = 9°$ corresponding to 180 mm. on scale.

The gauge was baked out for one hour at 330° C. and observations then taken on both the McLeod gauge and the molecular gauge under different conditions.

TABLE III.

Press. in McLeod.	D	Press. in Mol. Gage (Calc.).	Remarks.
0.49 bar	23°.7	2.6 bars	Liquid air on trap. Pump not exhausting.
0.8	105.	11.2	Removed liquid air.
0.27	36.4 mm.	0.27	Liquid air on trap and pump exhausting.
1.33	84°	9.3	Removed liquid air.
0.27	36.4	0.27	Liquid air on trap.
0.033	4.5	0.033	Liquid air on trap and pump exhausting.
Removed liquid air.		Stopped exhausting.	
	170 mm.	1.26	At end of 1 minute.
	100°	11.3	At end of 8 minutes.
Put on liquid air again			
	4.5 mm.	0.033	At end of 5 minutes.

It will be noted that the molecular gauge followed changes in pressure which were altogether lost as far as the McLeod was concerned. The

pressure of about 10 bar observed on removing the liquid air is evidently due to mercury vapor and non-condensible gases. Allowing about 1 bar for the pressure of the latter (indicated on McLeod) it follows that the pressure due to mercury vapor alone was about 9 bar at room temperature (298° Abs.).

According to Smith and Menzies[1] the vapor tension of mercury between 20° and 30° C. is as follows. (The pressure in bar was obtained by multiplying the pressure in mm. by $4/3 \times 1{,}000 \times \sqrt{200/28.8}$.)

Temperature.	Press. in Mm.	Press. in Bar.
20°	.0013	4.57
24	.00183	6.58
28	.00254	8.92
30	.00299	10.5

The determination of the vapor pressure of mercury as given above is in fair accord with the data for 28°–30° C.

A determination was also made of the vapor tension of ice at − 78° C. A bath of acetone with solid carbon dioxide was put around the liquid air trap and the pressure in the gauge measured while the pump was exhausting. The average of three determinations was 0.9 bar. Allowing for the difference in the temperature of gauge and liquid air trap, the pressure in the latter must have been $0.9\sqrt{195/298} = 0.78$ bar.

Extrapolating from the data given by Scheel and Heuse[2] for the vapor tension of ice at temperatures down to − 68° C., and allowing for the difference in molecular weight of air and water vapor, the pressure at − 78° C. is calculated to be about 0.2 bar. The pressure due to non-condensible gases was not over .05 bar in the above measurement.

3. *Calibration of the Gauge with Hydrogen.*—The theoretical conclusion that the indications of the gauge at constant pressure ought to vary with the square root of the molecular weight was tested by introducing hydrogen into the gauge instead of air.

The gauge used gave a deflection, with air, of 135 mm. per bar at 1,000 r.p.m. The sensitiveness with hydrogen should therefore have been $135 \times \sqrt{2/28.8} = 35$ mm. per bar at 1,000 r.p.m. The actual experiments gave values ranging from 37 to 42 mm.; the discrepancy being probably due to the presence of small quantities of air in the hydrogen used.

4. *Pressure in Tungsten Lamp.*—An ordinary 60-watt type Mazda lamp bulb was connected to a molecular gauge and after exhausting

[1] Jour. Am. Chem. Soc., *32*, 1447 (1910).

[2] Ann. Phys., *29*, 723 (1909).

them for one and a half hours at 250° C. they were sealed off and pressure observations taken at intervals during the life of the lamp.

The gauge used had a sensitiveness of 1,100 mm. per bar at 1,000 r.p.m. After sealing off and before lighting the filament, the pressure indicated was about 0.8 bar, but as soon as the filament was lighted the pressure decreased and inside of less than an hour it went down to below 10^{-3} bar. The lowest pressure was certainly well below 5×10^{-4} bar, and this determination may be regarded as an upper limit of the probable pressure in a tungsten lamp. It must be noted that the lamp used in these measurements was not given nearly as good a heat treatment as in the usual lamp exhaust.

The fact that the vacuum in a tungsten lamp improves when the filament is lighted has been known for some time, and the causes of this "clean-up" effect have been discussed in a number of papers published during the past three years by I. Langmuir.[1]

As the volume of the gauge was only slightly greater than that of the lamp bulb, we can conclude that the pressure obtained in a well-exhausted tungsten lamp, when the filament is lighted, is certainly well below 10^{-3} bar.

5. *Experiments with the Gaede Molecular Pump.*—A Gaede molecular pump[2] was run in series with an oil pump which in turn was connected to the "rough vacuum" line. A McLeod gauge was inserted between the oil pump and molecular pump in order to read the pressure on the rough side of the latter.

A liquid air trap was arranged between the gauge and the pump so that the diffusion of vapor of stopcock grease or of water could be prevented.

The following table shows the results obtained under different conditions:

No.	Press. on Rough Side of Molecular Pump.	Press. on Fine Side, Read by Molecular Gauge.	Conditions of Experiment.
1	13.3 bar	0.20 bar	After exhausting for 1 hour. No heating of gage; no liquid air.
2	13.3	0.09	Put on liquid air.
3	13.3	0.033	Heated gage to 300° C. for 1 hour, but did not heat glass tubing between gage and pump.
4	1,333	0.033	Let in dry air on rough side. Press. on fine side remained constant. Ratio = 40,000:1.
5	20,000	0.4	Let in more air on rough side. Ratio = 50,00:1.
6	20	$\ll$0.0007	Ratio 30,000:1.

[1] J. Am. Chem. Soc., *35*, 107 (1913), et sub.

[2] W. Gaede, Ann. Physik, *41*, 337 (1913).

It is evident from experiment (3) that heating the gauge alone was not sufficient to reduce the pressure in it owing to the constant diffusion of water-vapor from the tubing between the gauge and liquid air trap. But after this tubing had been also heated to 330° C. (experiment 6) the pressure in the gauge went down below 7×10^{-4} bar.

The sensitiveness of the gauge employed was such that 1 bar gave a deflection of 525 mm. at 1,000 r.p.m. At very low pressures a correction term had to be introduced for the eddy current effect in the framework of the mirror. The equation connecting pressure (p) and deflection (D) may be written in the form

$$D = \frac{525}{1000} p.r. + ki^2r, \tag{19}$$

where r = revolutions per minute, and k is a constant. See equation (18).

By noting D for different values of r and i, while p is maintained constant, it is possible to determine the value of k, and hence introduce the proper correction into the calculation for p. A special series of experiments showed that in the case of the above gauge, the equation for calculating p was of the form:

$$p = \frac{D}{0.525r} - 29 \times 10^{-5} i^2.$$

The value of i varied from 3 to 5 amperes. It is evident that the presence of this eddy current effect limited the sensitiveness of the gauge, for even at zero pressure, the deflection at 4 amperes and 10,000 r.p.m. would be 24 mm. At 7×10^{-4} bar, the deflection at 10,000 r.p.m. and 4 amperes would be 27.6 mm.

Under the best vacuum conditions, that is, using liquid air, and heating the gauge and all connecting parts to over 330° C. for about one and a half hours or longer, the deflections actually obtained were only slightly greater (1 or 2 mm. more) than the correction due to the eddy current effect. Allowing for experimental errors and for the difficulty in reading to an accuracy of 2 mm. when the disc was rotated at very high speeds, it is probably correct to conclude that the vacuum obtained was less than 7×10^{-4} bar. Assuming the ratio of 50,000 : 1 as holding down to the very highest vacuum conditions, the vacuum attained in the gauge with a pressure of 20 bar on the rough side should have been 4×10^{-4}. The experimental observations are in satisfactory agreement with this calculation.

It is worth noting in this connection that in his paper describing the construction of the molecular pump, Gaede states that (at 8,200 r.p.m.)

with a rough pump pressure of 1 mm. he obtained a pressure of .02 μ on the fine pump side corresponding to a ratio of 50,000 : 1. Both pressures were read by means of McLeod gauges.

OTHER VACUUM GAUGES.[1]

In this connection it might not be amiss to mention briefly some of the other vacuum gauges that have been suggested for the measurement of pressures below 1 bar.

1. The radiometer has been used by a large number of investigators. Dewar has stated the case for this instrument as follows:[2] "The radiometer may be used as an efficient instrument of research for the detection of small gas pressures. For quantitative measurements the torsion balance or bifilar suspension must be employed."

Some years ago Mr. W. E. Ruder, of this laboratory, developed a method of using the radiometer for the measurement of the gas pressure in incandescent lamps. "It was found that when exhausted to the degree required in an incandescent lamp the radiometer could not be made to revolve, even in the brightest sunlight. In order to get a measure of the vacuum, the radiometer vanes were revolved rapidly by shaking the lamp and the time required to come to a complete stop was therefore a measure of the resistance offered to the vanes by the gas, together with the frictional resistance of the bearing. The latter quantity was found to be so small in most cases that a direct comparison of the rates of decay of speed of the vanes gave a satisfactory measure of the degree of evacuation. In this manner a complete set of curves was obtained which showed the change in vacuum in an incandescent bulb during its whole life and under a variety of conditions of exhaust.

"The chief objections to this method of measuring vacua were the difficulty in calibrating the radiometer and the difference in frictional resistance offered by differnet radiometers. For *comparative* results, however, the method was entirely satisfactory."[3]

2. Scheele and Heuse[4] devised a manometer which has been used successfully for the accurate determination of the vapor pressures of mercury and ice at very low temperatures. This gage consists of two chambers separated by a copper membrane. One of the chambers is

[1] A good description of some of the gages mentioned in this summary is given in K. Jellinek's recently published "Lehrbuch der Physikalischen Chemie," I, 1.

Shortly after this paper was sent to the printer, a description of a modified Knudsen manometer was published by J. W. Woodrow, Phys. Rev. 4, 491 (1914). The sensitiveness of this gage is stated to be about 4×10^{-5} bar.

[2] Proc. Roy. Soc., *A*, *79*, 529 (1907).

[3] This account was kindly prepared by Mr. Ruder at the request of the writer.

[4] Ber. d. deutsch. phys. Ges., 1909, 1–13.

maintained at constant pressure while the other is connected to the system under investigation. The membrane presses on a glass plate and the variation in the thickness of the film is measured by noting the number of interference bands.

3. The McLeod gauge can be constructed so that it is sensitive to 0.01 bar. Its field of application is however necessarily limited.

4. Pirani[1] has suggested a resistance manometer which depends upon the fact that at low pressures the heat conductivity of gases is a function of the pressure. In consequence of this change in the heat conductivity, the apparent resistance of the wire changes with pressure of the gas surrounding it. The method has been improved by C. F. Hale[2] and the manometer has been found to give reliable results down to 0.00001 mm. of mercury, that is to about 0.01 bar.

5. Very recently W. Rohn has described a vacuum meter based on almost the same principles.[3] In this case the effect on the thermo-electromotive force of varying gas pressure is used as a method of determining very low gas pressures. The instrument is most sensitive between about 100 and 1 bar (0.075 mm. and 0.00075 mm. of mercury) the electromotive force varies approximately linearly with the logarithm of the pressure. At lower pressures, the sensitiveness diminishes quite rapidly.

6. Haber and Kerschbaum have used vibrating quartz fibers to measure the pressure of mercury and iodine.[4] This method was originally suggested by I. Langmuir[5] for measuring the residual gas pressure in sealed-off tungsten lamps and has been in use in this laboratory for about three or four years. As the pressure decreases the duration of the oscillations increases. Haber and Kerschbaum have deduced a relation between the pressure p and the interval t in which the vibration decreases to half its original amplitude, as follows:

$$\Sigma(p\sqrt{M}) + a = \frac{b}{t},$$

where a and b are constants and Σ denotes that the sum of the products of partial pressure and square root of the molecular weight is to be taken for each gas present. The lowest pressures actually measured by the above authors were about 0.015 bar, but the method has been used in this laboratory by Dr. Fonda to measure pressures considerably smaller than this.

[1] Ber. d. deutsch. physikal. Ges., 1906, 686.
[2] Trans. Am. Electrochem. Soc., *20*, 243 (1911).
[3] Z. f. Elektrochem., *20*, 539 (1914).
[4] Z. f. Elektrochem., *20*, 296, 1914.
[5] J. Am. Chem. Soc., *35*, 107, 1913.

7. W. Sutherland[1] and, subsequently, J. L. Hogg[2] derived simple relations between the pressure and the logarithmic decrement of a vibrating mica disc. The gauge based on this principle requires calibration at two known pressures, and is not very sensitive below about 0.01 bar.

8. M. Knudsen as the result of an elaborate study of the laws of "Molekularströmung" devised an absolute form of manometer[3] which depends upon the fact that at very high vacua there exists a very simple relation between the pressure and the torque imparted to a movable surface by the molecules flowing to it from a hotter surface. This relation has the form

$$K = \frac{p}{2}\left(\sqrt{\frac{T_1}{T_2}} - 1\right),$$

where K denotes the force of repulsion between two surfaces maintained at temperatures T_1 and T_2 respectively in a gas at pressure p. From the dimensions of the movable disc and the period of oscillation of the suspension, the value of K may be calculated and the gauge may therefore be used without any previous calibration. Knudsen uses this guage to indicate pressures as low as 2×10^{-3} bar.

9. Still more recently[4] Knudsen has devised a simplified form of vacuum guage, based on the same principles as the above, which he states to be sensitive to 2×10^{-4} bar.

CONCLUDING REMARKS.

The vacuum gauge described above might obviously be used to determine the magnitude of the "accommodation coefficient" for different gases, and thus test out the deductions advanced by different investigators.

Another line of investigation for which the gauge would be useful is the determination of the vapor tension of oils, waxes, etc., such as are used in connection with vacuum work. Owing to pressure of other work the writer has been prevented till now from carrying on such an investigation; but the results would be of great practical utility, as these materials are being constantly used by experimenters in connection with so-called "high vacuum" experiments.

In conclusion the author desires to express his appreciation of the

[1] Phil. Mag., *43*, 83 (1897).

[2] Proc. Am. Acad., *42*, 6 (1906); Phil. Mag., *19* (1906); Proc. Am. Acad., *45*, No. 1, Aug., 1909.

[3] Ann. Physik *32*, 809 (1910).

[4] Ann. Physik *44*, 525 (1914).

kindly interest shown by Dr. I. Langmuir during the progress of the investigation and for helpful suggestions.

SUMMARY.

A theoretical consideration of the behavior of gases at very low pressures shows that a rotating disc exerts a torque on a disc suspended symmetrically above it, that is proportional to the quantity $\Sigma(p\sqrt{M/RT})$. Here p denotes the partial pressure and M the molecular weight of each constituent present in the gas and R and T have their usual signification.

The paper contains the description of a vacuum gauge based upon this principle, and also the results of a number of measurements carried out with its aid.

It was found that in order to obtain the best possible results with a Gaede molecular pump, it is necessary not only to heat the vessel to be exhausted and connecting tubing to a temperature at which most of the moisture adsorbed in the walls is driven out, but also to insert a liquid air trap to prevent the diffusion backwards of condensible vapors.

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FLICKER PHOTOMETER MEASUREMENTS BY A LARGE GROUP OF OBSERVERS ON A MONOCHROMATIC GREEN SOLUTION.

BY HERBERT E. IVES AND E. F. KINGSBURY.

A DETERMINATION of the mechanical equivalent of light, now in progress, demanded the photometric evaluation of the monochromatic green radiation of the mercury arc, wave-length .5461 μ. Since in measuring the luminous intensity of a colored light in terms of the present yellowish white standards it is always necessary, no matter what method of visual photometry is employed, to use a large number of observers, some means was necessary to determine that value by a separate experiment. It is well known that the current-intensity relation in the mercury arc is not nearly reliable enough to permit of holding the radiation constant by maintaining the current at a fixed value. It was therefore necessary that our measurements be made on some auxiliary standard which would possess maintainable and reproducible characteristics. For this standard we decided upon an absorbing solution, which should be used over the standard "4-watt" carbon lamp, to be of such constitution that the transmitted light would be a visual match with the green mercury light. Such a solution was developed and used. The details of its use and the results of its measurement have points of interest apart from the question of the mechanical equivalent of light and are, therefore, presented here separately.

THE ABSORBING SOLUTION.

The solution, which transmits light exactly matching the green line .5461 μ when used over a standard carbon lamp, is a mixture of potassium dichromate and cupric chloride, together with a sufficient amount of nitric acid to hold the two in solution. The composition for 25 millimeters thickness is:

Potassium dichromate	2.5 grams.
Cupric chloride	265.0 grams.
Nitric acid (sp. gr. 1.05)	26.5 cu. cms.
Water to one liter.	

The cupric chloride before weighing is dried by being raised to a tem-

perature of 50° centigrade. When mixed the solution is filtered through a triple paper filter.

Careful photometric tests have shown that with these precautions the solution is absolutely reproducible and, certainly over periods of several weeks and probably much longer, shows no detectable fading or other change.

In the use of the solution it is necessary to know its change of trans-

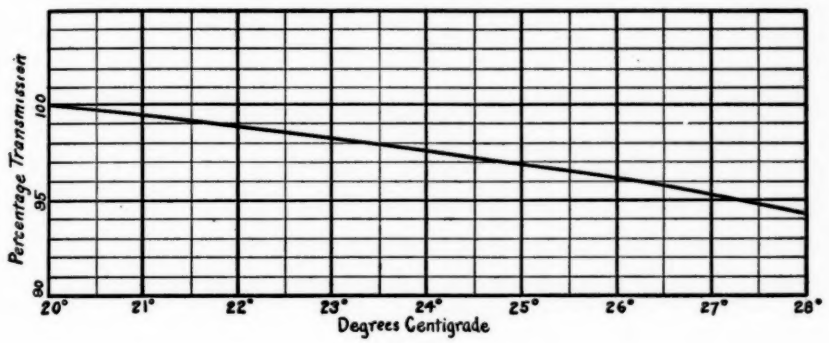

Fig. 1.

Temperature Coefficient of Transmission of Green Solution.

mission with change of temperature. This has been determined by comparison on the photometer of the light through a solution at the standard temperature of 20 degrees centigrade with that through another at various higher and lower temperatures. The values are shown in Fig. 1.

The Absorption Cells.

Great emphasis must be placed on the importance of using a reliable type of absorption cell. In a recent investigation we have found differences of as much as five per cent. in the transmission of supposedly similar clear glass tanks. As a result of our study we have come to the use of a special type of tank, shown in Fig. 2. The solid glass frame is exactly 25 millimeters thick, in this case, and can be obtained without difficulty from a good glass worker. The two removable faces must be selected with the greatest care, for upon them depends the value of the

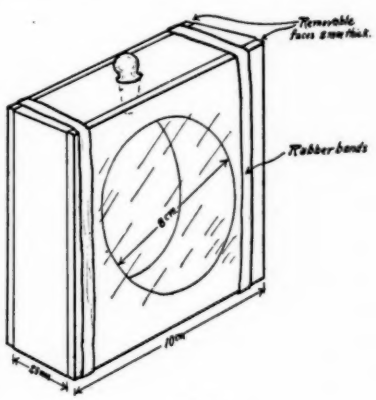

Fig. 2.

Type of Absorption Cell Used.

cells' transmission. We are using plates two millimeters thick, of special clear white glass which shows no color when viewed edgewise. In addition, we have tested these glasses through the spectrum by means of approximately monochromatic color screens on an ordinary Lummer-Brodhun contrast photometer, finding no selective absorption.

The cleaning and handling of these faces is also of importance. Whenever removed they are washed with hot water and soap, rinsed thoroughly with hot water and wiped carefully with a clean soft towel free from grit. After cleaning they are laid against the glass frame, held in place with rubber bands, and a seal of paraffine run around the edges with a hot metal spoon to prevent the cell from leaking. The solution, or the clear water as the case may be, is introduced through the stoppered opening at the top.

In order that the necessary cleaning and the inevitable process of surface decay shall not introduce progressive and unnoticed differences between the cells used they should be periodically compared for their total transmission when filled with clear water, and if it becomes necessary, new faces should be obtained.

THE METHOD AND DETAILS OF THE MEASUREMENT.

The transmission of this solution was measured by means of the special flicker photometer recently described in the PHYSICAL REVIEW.[1]

The conditions of measurement were those determined upon as a result of the extended study of colored light photometry by one of the present writers, involving the maintenance of one field brightness throughout, the use of a small field and other details for which reference may be made to the work quoted.[2]

The experimental procedure was as follows: The flicker photometer was mounted at one end of a three-meter photometer bar. The carbon lamp, a 100-candle-power point source of the stereopticon type, carefully matched in color with a standard supplied by the Bureau of Standards, was upon a movable carriage. The absorption cells, one containing clear water, the other the green solution, were held in supports before the photometer. The first measurement was upon the transmission of the clear water. The lamp was set at such a distance from the photometer screen that the brightness of the photometer field was that of a white surface illuminated by 25 meter-candles. After five settings were made with the clear solution it was replaced by the green one, and the lamp moved to a nearer position so that the new measurement was carried

[1] PHYSICAL REVIEW, October, 1914.

[2] Photometry of Lights of Different Colors, Phil. Mag., July–December, 1912.

out at approximately the same field brightness. Five settings were made on this and then the process was repeated so that each observer made ten settings. Between groups of settings the green solution was immersed in a basin of water held at 20 degrees centigrade, on removal from which it was wiped dry with a clean soft towel. From the relative distances of the lamp, correcting for the thickness of glass and water, the transmission of the solution, compared with clear water, is obtainable.

This description refers to a single observer. In order that the result may apply to the average eye, it is necessary to secure values from a large number, just how large a number being one of the points to be determined by the investigation. The measurements here recorded were on some sixty-one observers, members of our laboratory force employees of other departments of the company, and some visitors. Many of the subjects were new to photometric reading, the majority had never read a flicker photometer. The determinations extended over a period of about two months.

No other tests of color vision were made so that it is possible we have among the number some whose vision might be classed as abnormal. We are reasonably sure that none is included who would fall in the classification of "color blind," because measurements made on one known color-blind person gave a result 60 per cent. larger than the mean, falling in a class entirely apart. The observers probably constitute an "average" group of men and, as will be seen, the number is apparently large enough so that the addition or subtraction of any chance observer or any group chosen at random would leave the mean value substantially the same.

RESULTS.

In Fig. 3 are plotted the results obtained by the observers in the order taken, together with the mean of all values up to each point. The final mean value for the transmission of this solution at 20 degrees centigrade, compared with clear water, is .0437. Individuals vary from this as much as sixteen per cent. above, and twenty-nine per cent. below (leaving out of account the color-blind observer above mentioned).

DISCUSSION.

The question of the precision and reproducibility of measurements of this kind has been discussed quite fully in the previous papers. The present measurements merely confirm the previous findings. Repeat measurements were made on some of the observers, with results checking to about one per cent. About a third of the observers had previously taken measurements on a similar but ultimately unsatisfactory solution.

Their relative positions with respect to the mean were the same on that series and on this with only a few slight changes in order, showing that no considerable changes in an observer's criterion are to be looked for.

We have searched in vain for any connection between the position of an observer in this series and any of his physical characteristics, such as age.

A point of extreme importance in colored-light photometry is here demonstrated in very convincing form. This is that precision in reading and agreement between different observers of a small group are no

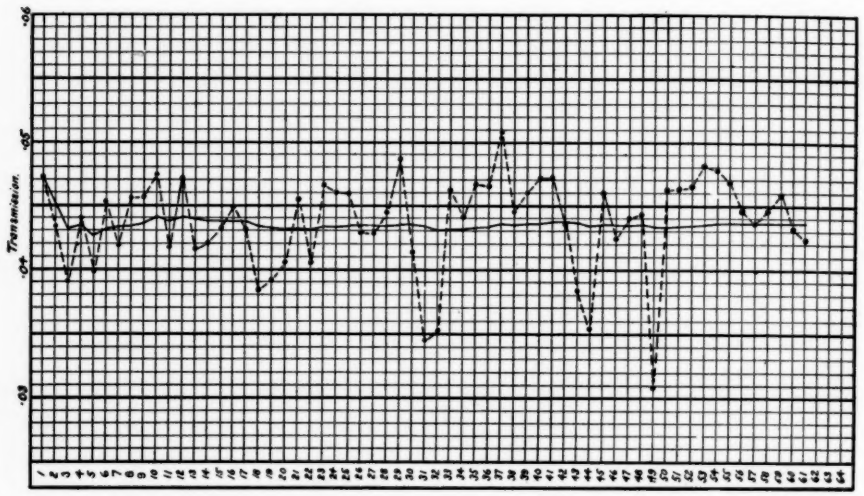

Fig. 3.

Individual Readings on Transmission of Green Solution.
Points,—observations.
Numbers,—observers.
Solid line,—mean of all observations up to and including point.

evidence whatever of accuracy. The emphasis placed upon this is prompted by the widely-found belief that if one observer reads with a small mean variation and consistently from time to time it is evidence that his readings are "right"; and the similar idea that if several observers who happen to be associated together read alike, their result must be "right." This conclusion is valid only where there is no color difference. Where there is a color difference, only some such investigation as that here reported can be established what is to be taken as correct. The correctness of an individual observer's setting has nothing whatever to do with his skill or training—it is dependent upon the color sensibility of his retina. The agreement of several observers is a matter of chance, except where, using the equality of brightness or direct comparison

method, observers have consciously or unconsciously altered their criteria toward a common mean. The justice of this criticism is strikingly illustrated by the fact that observers 51, 52, 53, 54 and 55 constitute the entire group of photometrists from a laboratory in which measurements involving a considerable color difference were part of the regular routine. They agree with each other to within about three per cent. but their mean on this particular color difference is eight per cent. above the average for all.

These considerations indicate the importance of a definite scheme of selecting observers from a group of at least twenty-five or thirty where lights of different color are to be evaluated. They also emphasize the desirability of methods of colored-light photometry by which individual observers may secure the results of the average eye as established by measurements on a large group. We expect to report shortly on means for achieving this end.

PHYSICAL LABORATORY, THE UNITED GAS IMPROVEMENT COMPANY,
PHILADELPHIA, November, 1914.

ISOLATED CRYSTALS OF SELENIUM OF THE SECOND AND FIFTH SYSTEMS, AND THE PHYSICAL CONDITIONS DETERMINING THEIR PRODUCTION.

BY F. C. BROWN.

RECENTLY[1] I described new crystal forms of metallic selenium, in connection with their optical and photo-electrical properties. One form was definitely classified as belonging to the rhombohedral hexagonal system, but because of the smallness and incompleteness of the lamellar crystals they could not be classified. Since then we have succeeded in obtaining lamellar crystals well developed on all surfaces, so that the angles between all faces could be determined with a certain accuracy, by the use of a circle goniometer.

The reproduction in Fig. 1 is that of lamellar crystals belonging to the fifth or monoclinic system. A diagram of one of the crystals is given in Fig. 2, together with the plane of symmetry and the axes. The surfaces m, n and O are at right angles. The ratios of the axes of one crystal were about $a : \bar{b} : c = 1.0 : 0.18 : 6.0$, $\beta = 60°$. The longest crystal was about 10 mm.

Crystals of the hexagonal system were photographed and reproduced in the previous article referred to. In the acicular crystals the angles between the faces were all 60°. Other crystal forms and growths were reproduced but thus far it has not been possible to definitely classify them.

Crystals of both systems were produced by sublimating amorphous selenium after it had been melted. In general the higher the temperature at which the crystals were formed from the vapor, the more certainly would the hexagonal needles form. At temperatures between 190° and 220° large quantities of the hexagonal needles were obtained, and at the highest temperature various undetermined forms and combinations of forms appeared. It required one or two weeks for the crystals to form in the larger sizes.

By keeping the oven temperature at 170° ± 1° for a period of three months the monoclinic crystals shown in Fig. 1 were made. The selenium vapor was driven from the amorphous form at a somewhat higher temperature and condensed on the crystal at the above temperature. These

[1] PHYS. REV., N. S., Vol. 4, p. 85, 1914.

Fig. 1.

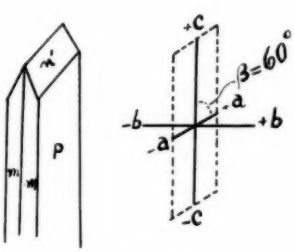

Fig. 2.

F. C. BROWN.

crystals were formed in a vacuum, except for the selenium vapor present. It was observed that these lamellar crystals would also form at temperatures as high as 190° in an atmosphere of air at normal pressure. Also it seemed necessary to increase the temperature necessary for the formation of the hexagonal form when the vapor was deposited in air at normal pressure. These observations lead to the supposition that the vapor density as well as the temperature is a determining factor in the control of the crystal forms.

As far as has been tested the electrical, the light-electric, and the electro-mechanical properties of the selenium crystals of the two systems do not show any distinct differences. The value of the conductivity, and the sensibility to light in different parts of the spectrum are of the same order of magnitude. The above facts indicate that the above properties are seated in the molecular structure, and that the minor variations are controlled by the crystal structure.

It has been observed that the light on going through the wedge-shaped end of a lamellar crystal is deviated about 30°. Since we know the angle of the wedge, we have a direct method of measuring the index of refraction of a conducting substance. This will be carried out at the earliest opportunity.[1]

UNIVERSITY OF IOWA.

[1] Since writing this article I have obtained the red crystalline variety of selenium by sublimation. These crystals are doubly refracting and are no doubt the same as those obtained by various chemical means. The precise physical conditions *i. e.*, vapor pressure and temperature governing the formation of the red crystals by sublimation are yet to be determined.

THE INTENSITIES OF X-RAY SPECTRA.

BY DAVID L. WEBSTER.

ACCORDING to Bragg[1] if X-rays of a given frequency are reflected from a crystal at the angle θ, the intensity of the reflection may be expressed within the limits of experimental error by the equation

$$I_\theta = \frac{A(1 + \cos^2 2\theta)}{\sin^2 \theta} e^{-B \sin^2 \theta}. \quad (1)$$

The constant A depends on the kinds and numbers of atoms in the crystal and the intensity and frequency of the incident rays, while B depends on the thermal properties of the crystal. The exponential factor has been found by Debije[2] to result from displacements of the atoms from their normal positions by heat vibrations. The factor $(1 + \cos^2 2\theta)$ originally deduced by Thomson[3] for any kind of secondary X-radiation, results from the fact that the component of vibration of the electron perpendicular to the plane of the incident and reflected beams will make it radiate equally in all directions in that plane, while the other component, being perpendicular to the incident beam but not to the reflected one, will make the amplitude of the secondary radiation vary as the cosine of the angle between the beams, that is, as $\cos 2\theta$. Adding the squares of the amplitudes, one obtains the factor $(1 + \cos^2 2\theta)$ in the expression for the reflected energy. This factor suggests some interesting experimental possibilities, which will be discussed below.

The factor $\sin^2 \theta$ in the denominator, as Professor Bragg has recently told me, has not yet been accounted for. The object of this paper is therefore to account for it, and to show that it signifies that the train of waves emitted by an atom in the radiator is short compared to the distance the rays travel through the crystal. This, as a matter of fact, does not seem very unreasonable when we consider the origin of the rays, in the impacts of cathode particles against these atoms. The possibility of a further experimental test of this result is discussed below.

Reflection of a Long Train of Waves.—Since Debije[4] has found that

[1] W. H. Bragg, Phil. Mag., *27*, 895, 1914.
[2] Verh. d. Deut. Phys. Ges., 15, Nos. 15 and 16, 1913.
[3] J. J. Thomson, The Conduction of Electricity through Gases.
[4] L. c.

the heat motion of the atoms should not affect the sharpness or position of the reflection, and should change the intensity only by the factor $e^{-B \sin^2\theta}$ we shall treat the atoms as though they were stationary, and insert this factor at the end. We shall also consider only the vibration components perpendicular to the plane of incidence, and insert the factor $(1 + \cos^2 2\theta)$ in the final expression for the reflected energy. We shall consider first the case of reflection of a train of waves from a single layer of atoms at an arbitrary angle θ, and then take account of the influence of the different layers upon each other, treating the case of a very long train first, and then that of a short one.

In the accompanying figure, S represents the source of rays, S' its

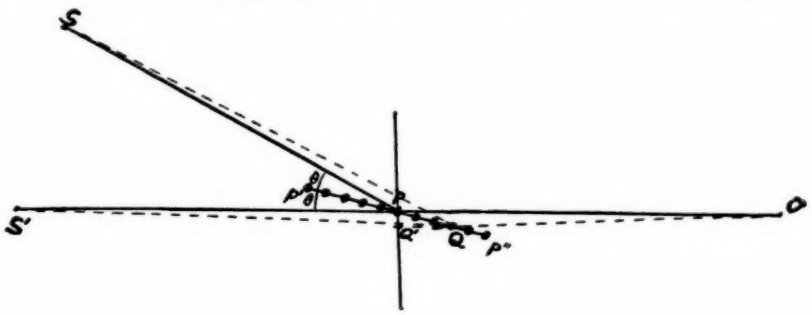

Fig. 1.

image in the plane $P'P''$, containing the atoms, and O the observing point. To find the intensity of the reflected beam, we must take account of the components of electric force from each atom in the plane, paying especial attention to their phases. This may be accomplished best by treating the waves as though they came from the image S' and were re-radiated from the various atoms, but not otherwise propagated through the plane.

For the radiation from the atom Q, whose distance from P is small compared to PO or PS, we may find the phase retardation, relative to that from P, as

$$2\pi \frac{\overline{S'Q} + \overline{QO} - \overline{S'O}}{\lambda},$$

λ being the wave-length of the rays. This evidently is the same, except for small quantities of a higher order, as if Q were replaced by its projection, Q', on the plane through P perpendicular to PO. Projecting all the positions of the atoms in this way, we obtain on the plane PQ' a uniform distribution of points, of a surface density $Na/\sin\theta$, where N is the volume density and a the distance between the planes.

Thus the problem resolves itself into one almost exactly similar to

the application of Huygens's principle of light to the case of a spherical wave spreading from the image S'. For it is well known, that for cases where $S'P$ and PO are large compared to the wave-length, an approximate form of this principle applied to the plane PQ' is obtained by supposing the ether to absorb and re-radiate the energy coming to it just as the atoms do here. This, in fact, is the form in which the principle is usually stated in textbooks on optics. In this case, however, the effect of any surface element dS in the plane PQ' is the electric force radiated from the atoms, $NadS/\sin\theta$ in number, whose projections lie in this element. Therefore, since the distance between atoms (about 10^{-8} cm.) is small compared to the radius of the first quarter wave zone (10^{-3} to 10^{-2} cm.), and this in turn is small compared to the distances to S' and O, we may say without further computation that the reflected amplitude at O will be proportional to the surface density, $Na/\sin\theta$, of the projections of the atoms on the plane PQ'. For this result it is not necessary to assume that θ is an angle of reflection for the crystal as a whole.

Thus if the amplitude of the electric force of the incident wave is E_i, that of the reflected one from a single layer will be

$$qE_i\frac{Na}{\sin\theta}, \tag{2}$$

where q is a coefficient expressing the reflecting power per unit surface density of the atoms, and depending on their weight and structure.

If the reflections from successive layers are not retarded by an integral number of wave-lengths, they will of course cancel each other; but if they are thus retarded, they will reenforce. This leads to the well-known equation,

$$n\lambda = 2a\sin\theta, \tag{3}$$

for the value of θ giving the reflection of order n and wave-length λ.

Since the rays are absorbed in entering the crystal to the point P, we have

$$E_i = E_0e^{-kx}, \tag{4}$$

where x is the distance the incident wave has come through the crystal; and since the reflected wave is absorbed in the same ratio before it emerges, its amplitude at the point O is

$$qE_0\frac{Na}{\sin\theta}e^{-2kx}. \tag{5}$$

In the case of reflection at the proper angle, we must now sum these amplitudes from all the planes. Replacing x by $y/\sin\theta$ so that y is the

depth of the plane from the face of the crystal, and replacing a by dy, we may turn the summation into the integral,

$$qE_0 \frac{N}{\sin\theta} \int_0^Y e^{-\frac{2ky}{\sin\theta}} dy = \frac{qE_0 N}{2k} (1 - e^{-\frac{2kY}{\sin\theta}}). \qquad (6)$$

Therefore, if kY is large, as it generally is, this is independent of θ, and long trains of waves will not give the $\sin^2\theta$ in the denominator of expression (1).

If the crystal is composed of small pieces, each acting independently, then we might assume that the rays would go through one piece with no appreciable loss, and would therefore be reflected from it in proportion to $1/\sin^2\theta$. This may be seen from the fact that the number of reenforcing layers would then be constant, and equal to the whole number in the uniform piece of the crystal.

Here, however, we should have to remember that the rays would still penetrate with a given intensity to a depth proportional to $\sin\theta$, so that the number of pieces of the crystal that reflect would vary in that way. Since they would all reflect independently, we should have to add reflected energies, rather than amplitudes, and the resulting energy would be proportional to $1/(\sin\theta)$ rather than to the $1/(\sin^2\theta)$ given by experiment. Thus it appears that the long train hypothesis must be abandoned.

This result has been obtained also by Darwin,[1] but the calculation is given here because it is of interest in what follows.

Short Trains of Waves.—Turning now to the opposite hypothesis, that the trains of waves are short compared to the distance they go through the crystal, and until further notice considering the crystal uniform, we may make use again of the expression (2) for the amplitude reflected from a single plane. In this case, however, the calculation of the number of reënforcing planes is more complicated.

For simplicity, let us consider a single train, of M waves, reflecting in the nth order. At the instant when the nth wave of the reflected train from a given layer of atoms arrives at the point O, the wave arriving from the next layer above is the $2n$th, and the one from the layer above that is the $3n$th, and so on. Thus the total number of reënforcing layers is not M, but M/n; and neglecting the change of amplitude of the incident wave by absorption in going the length of the train, the whole reflected amplitude is

$$\frac{M}{n} qE_i \frac{Na}{\sin\theta} e^{-kx},$$

[1] C. G. Darwin, Phil. Mag., *27*, 675, 1914.

where e^{-kz} is the fraction that is not absorbed before leaving the crystal, and E_i itself is E_0e^{-kz}. Thus, for a given value of x while the reflected train is passing O, its amplitude will be proportional to $a/(n \sin \theta)$, or to $1/(\sin^2 \theta)$, so that the energy density in it varies as $1/(\sin^4 \theta)$.

The length of this train, however, will depend, not on that of the incident one, but on the distance it penetrates into the crystal and the relative retardation of the waves from the surface and from the lower layers. Therefore, to find the whole reflected energy, we must evaluate the energy density at a given distance from the front of the reflected train, and integrate over its whole length. Since the mean energy density over a whole wave is $1/8\pi$ times the square of the amplitude of the electric vibration, and the retardation of the part of the train reflected from the depth y (that is, the distance from the front of the reflected train) is $2y \sin \theta$, the whole energy is

$$\frac{1}{8\pi}\left(\frac{M}{n}qE_0\frac{Na}{\sin\theta}\right)^2\int_0^\infty e^{-\frac{4ky}{\sin\theta}}\cdot 2\sin\theta\, dy = \frac{1}{16\pi k}\left(\frac{M}{n}qE_0Na\right)^2.$$

Since $n\lambda = 2a \sin \theta$, we may write this as

$$\frac{M^2\lambda^2q^2E_0^2N^2}{64\pi k \sin^2\theta}, \tag{7}$$

thus showing that the resultant energy from any short train is proportional to $1/(\sin^2 \theta)$ for all faces of the crystal that have uniform distances between the planes. Moreover, since a does not appear in (7), this explains the fact that this proportionality holds whether the distances are the same for different faces or not.

To improve this expression (7), let us note that $\frac{1}{8\pi}E_0^2L$ is the intensity I, of the incident train, and $M\lambda$ is the whole length, L, of the wave train. Introducing these substitutions and the factors $(1 + \cos^2 2\theta)$ and $e^{-B\sin^2\theta}$ mentioned above, we have as a final expression for the reflected energy

$$\frac{Iq^2LN^2(1+\cos^2 2\theta)}{8k\sin^2\theta}e^{-B\sin^2\theta}. \tag{8}$$

This shows that in Bragg's formula

$$A = \frac{IqL^2N^2}{8k}. \tag{9}$$

Since according to the short train hypothesis, there is no reënforcement of waves from very different depths, the assumption made above, that the crystal is uniform, is not essential if the uniform pieces within it are large compared to the lengths of the trains of waves. Thus (8) is really independent of the uniformity of the crystal.

Since (8), rather than (6), is verified by experiment, we may conclude that the trains of waves are really short compared to the distances they penetrate into the crystal, or, if the crystal is not uniform, they must be short compared to the irregularities in its structure.

A further experimental test of this short train hypothesis should be found in measurements of the reflecting power of a crystal for rays which have already been reflected once. Such waves, being in long trains, should show less difference in the intensities of different orders than in the first reflection, provided that the first crystal is really uniform enough to give a long train of reflected waves. Moreover, under these conditions, the larger value of L in a train that has already been reflected once should make the reflection coefficient greater in the second reflection than in the first. The apparatus to test these points is now being constructed.

The Polarization of Reflected X-Rays.—Another point to be tested is the polarization of the reflected rays that is assumed in deriving the factor $(1 + \cos^2 2\theta)$ in equation (1). This factor, as we have seen, is due to the fact that in the plane of incidence each atom scatters the component of the electric vibration perpendicular to that plane equally well in all directions, while the other component is diminished at the angle 2θ from the original beam, by the ratio $\cos 2\theta$. Thus the intensities of the two components will be in the ratio of 1 to $\cos^2 2\theta$.

If now the beam is reflected twice in a horizontal plane, the ratio of the intensity of the vertical component to that of the horizontal one after the second reflection will be 1 to $\cos^4 2\theta$, and the intensity of the resulting beam will bear the ratio $(1 + \cos^4 2\theta)$ to that of the vertical component, which we may take as the standard. If, on the other hand, the second reflection is in a vertical plane, the vertical component will then be diminished, so that the intensity of each will be the fraction $\cos^2 2\theta$ of the above standard, and the whole intensity will be $2 \cos^2 2\theta$.

If now θ is as large as 20°, as in some of Bragg's experiments, the ratio of the total intensities in these two cases will be 1.171 to 0.826. At larger angles, if the intensity is not too weak, such measurements should show an even greater difference, and at $\theta = 45°$, the vertical reflection would be impossible, as the beam would then be totally plane-polarized.

SOME SECONDARY EFFECTS FROM ROENTGEN RAYS.

BY PAUL T. WEEKS.

IN an article in Science (38: 347–8) F. R. Gorton describes an anomalous effect obtained with Roentgen rays. A photographic plate was laid film side down on one piece of metal with a second piece of metal above the plate so that the areas of contact of metal and plate overlapped. When exposed to Roentgen rays from above and developed the plate was much less dense where the two pieces of metal overlapped than where the plate was in contact with the upper piece of metal alone. The plate was densest where in contact with the metal below alone. Mr. Gorton found this effect with Ag, Au and Cu, but not with Pb.

An arrangement similar to that described by Mr. Gorton was tried by the writer with Ag, Pb, Fe, Cu and other metals and in every case the effect found by him in the case of Ag was observed. In an attempt to find an explanation for this effect, further experiments were carried out as described below.

Exposures were made from above with the plate face down, part of the plate being in contact with a piece of metal below the plate and part protected by a piece of metal a few millimeters below the plate. The second piece of metal was found to produce nearly the same effect as the first. In fact it was found that nearly as much effect was obtained when there was no metal directly behind the plate if only the plate was protected from a general scattered radiation from below. Thus a plate which was exposed with face down on the end of an iron tube four or five inches long showed 18 per cent. more light transmission through the portion over the inside of the tube than through the portion outside the tube, while the portion in contact with the iron was darkest of all. Plates exposed through an opening in a lead screen which would prevent a general scattered radiation from behind were in almost every case darker over the metal.

These experiments indicate two effects from metal behind the plate; first a protection of the plate from a scattered radiation originating in the region behind the plate; second, the production of a secondary radiation in the metal itself which counteracts this protective effect, the ratio between the general secondary radiation and the radiation from

the metal itself determining which effect will predominate. These two effects together explain the phenomenon as originally observed.

The next step was to compare the effects from different metals with primary rays of various penetrations. First exposures were made with strips of Pb, Fe, and Cu under the plate and similar strips laid at right angles to these above the plate. At tube voltages from 55 to 90 K. V. the plate was lightest over the Fe and darker over the Cu and Pb in order without regard to the metal above. A series of exposures were next made with strips of different metals behind the plate, the plate being exposed behind an opening in a lead screen. At 90 K. V. tube voltage the darkening was in this order, Ag most, Pb, Fe, Al, one plate for instance giving the following values of relative transmission; over Ag 75, Pb 92, Fe 94, Al 100. At 55 K. V. and 35 K. V. the order was the same, one plate at 35 K. V. giving the following values of relative transmission; over Ag 85, Pb 89, Fe 95, Al 100. At 25 K. V. the order was changed, a typical plate giving these values; Pb 83, Fe 84, Ag 89, Al 100. At 17 K. V. and 14 K. V. the Ag gives very much less darkening and the Fe much more. For instance a typical plate at 14 K. V. gave these values: Zn 39, Cu 55, Fe 61, Pb 63, Ag 80. At all voltages tried Zn, Cu and Fe produced darkening in the order named.

These results may be explained from the relative abilities of the metals to emit secondary radiation under excitation by primary rays of different penetrations. The curves in Fig. 1 are drawn from data given by Sadler,[1] and by Whiddington and Kaye,[2] to show the relative secondary emission

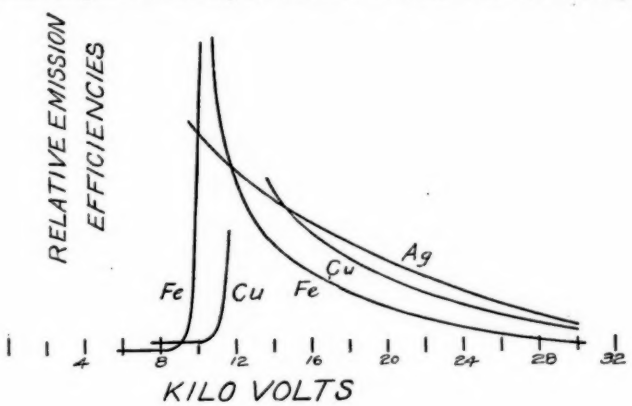

Fig. 1.

efficiencies of these metals for primary radiations corresponding to different tube voltages. The curve for Al is not shown as it would be a

[1] Phil. Mag., 22: 541.

[2] Kaye's X-Rays, p. 127.

nearly straight line lying very close to the axis, corresponding to very low emission efficiency at all voltages and consequently very slight darkening effect. The curves explain the order of darkening from Fe, Cu and by analogy that from Zn. The fact that silver produced relatively slight darkening at low voltages may perhaps be explained from the fact that the absorption coefficient of the silver of the emulsion is very much greater for the characteristic radiation from the Fe, Cu and Zu than for the scattered radiation from the Ag. However the darkening effect even from the silver was greatly reduced by interposing a single thickness of paper between the metal and the plate, which would seem to indicate that much of the darkening effect was produced by a corpuscular radiation, easily absorbed. The explanation of the sudden increase in the darkening from the silver at about 25 K. V. is undoubtedly that the characteristic radiation of silver was excited at that voltage, the total amount of secondary radiation from silver being thereby greatly increased.

The effects observed are interesting in connection with the suggested use of a metal behind the plate as an intensifying screen. Figs. 2 and 3 show some results obtained when using as object a paraffine block made

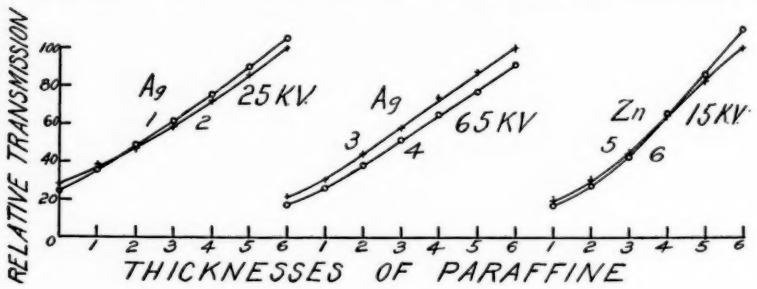

Fig. 2.

Curves with circles are with metal backing. Curves with crosses are without metal backing.

in the form of steps 1 cm. high. The curves show the relative light transmission through different parts of the plates under successive thicknesses of paraffine with and without metal behind.

The plate from which the values for Curves 3 and 4 were obtained was exposed through an opening in a lead screen; in exposing the other plates no attempt was made to reduce the scattered radiation from behind. The values for curves 11 and 12 were obtained from a plate exposed behind successive thicknesses of sheet iron instead of paraffine. Curves 3 and 4 indicate for a particular case the amount of intensification obtained. Although marked in many cases this effect can hardly be compared to that obtained with a regular intensifying screen.

The most noticeable effect is shown by the other curves. In these the steeper slope indicates a better contrast in the portion of the plate backed by the metal. This was also shown in a plate exposed under a block of paraffine on which were laid objects of varying thickness. The rela-

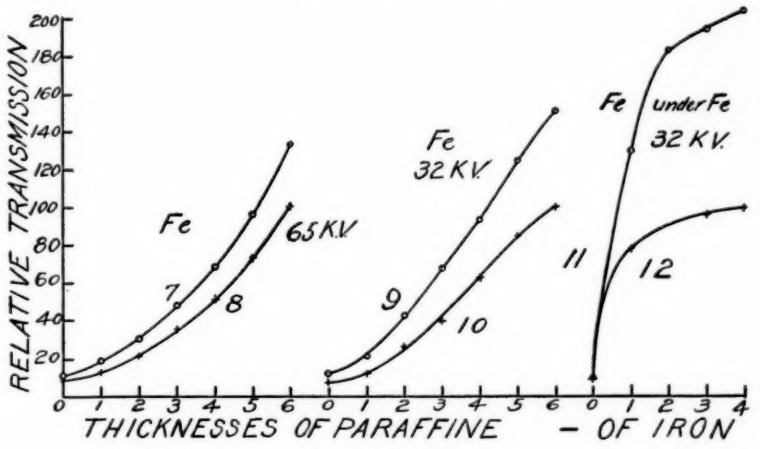

Fig. 3.

Curves with circles are with metal backing. Curves with crosses are without metal backing.

tive darkening under corresponding parts where the plate was backed by metal and where unprotected is shown by the relative transmissions. This improvement in the contrast of the plate is to be expected as a

	Metal Backed.	Unprotected.
Glass tube	21	20
	36	31
	32	28
Rubber tube	20.5	19
	24	21.5
	23.5	21
Rubber tube	33	24.5
	26	21.5
	2	3
Metal	35	32.5
	80	68
	78	67

result of preventing the fog due to secondary scattered radiation. These plates were exposed under varying conditions so that the results obtained can not be taken as typical for the metals used.

These last results show plainly the advantage to be obtained from a

reduction of the amount of scattered radiation originating behind the plate. If the amount of this is sufficiently reduced the use of metal behind the plate becomes of relatively small advantage at best. But if provision is not made for reducing this scattered radiation from behind the plate the use of metal backing results in much better contrast.

It is also evident that a sensitive plate in the neighborhod of an active Roentgen ray tube should be entirely surrounded by lead and not merely screened from the direct radiation from the tube.

A mechanically rectified high-tension alternating current was used for operating the tube. The voltage was measured by means of a Siemens-Halske attracted disk type voltmeter. The values of voltage used were obtained by multiplying the voltmeter readings by 1.4 to give the peak values of the voltage.

SUMMARY.

1. An explanation is given for a peculiar effect from secondary Roentgen radiation, originally noted by F. R. Gorton.

2. The relative intensifying effect from different metals placed behind a plate when exposed to Roentgen rays of various penetrations has been investigated.

3. Additional evidence is given of the need for preventing or screening off the general scattered radiation.

I wish to acknowledge the suggestions and assistance given me by Professor J. S. Shearer.

PHYSICAL LABORATORY,
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A NEW HYDROMETER OF TOTAL IMMERSION WITH ELECTRO-MAGNETIC COMPENSATION.

By Anders Ångström.

IN the present note, I intend to propose a new method for determining the specific gravity of liquids and especially the specific gravity of seawater, from which the salinity may be calculated with high degree of accuracy.

The different hydrometers which have been used or merely suggested for this kind of work may be divided into two groups: floating hydrometers and such as are worked by some method of total immersion. As regards the floating hydrometers, it may be mentioned that Nansen has made a very careful study of the errors and irregularities adhering to these instruments. Nansen arrives at the conclusion that the chief cause of their drawbacks lies in uncontrollable changes in the surface-tension of the water; he also discusses the precautions and corrections that are necessary in order to eliminate these influences. The number of the precautions to take is however so large, that several eminent oceanographers are inclined to discard the floating hydrometer as unsuitable for accurate work.

In order to avoid the errors arising from variations in the surface tension of the liquid, hydrometers of *total immersion* have been invented; and here I may refer to the papers of Guglielmo, of Reggiani and especially to the discussion presented by Nansen, who gives several references regarding this subject. It appears that the methods of total immersion are primarily of two kinds: such methods, where the hydrometer itself through the adding of small loads is brought to the point where it just sinks down totally in the liquid, and such ones where the specific gravity of the liquid is changed, in order to attain the same end. The ingenious method used by Nansen is a combination of these two principles. Very small changes in the density can be produced by raising or lowering the temperature of the liquid by means of a thin platinum spiral, heated by an electric current.

The objections against these methods, that, when properly used, seem to be very accurate, arise from the inconvenience of loading and unloading the hydrometer, or, if the heating method is used, from the difficulty of

knowing if the hydrometer and the air contained in it have reached the temperature of the surroundings or not. And finally the hydrometer must then be adjusted to an unstable equilibrium, as the least change will bring the hydrometer to the surface or to the bottom of the vessel. A considerable time is therefore required for each determination.

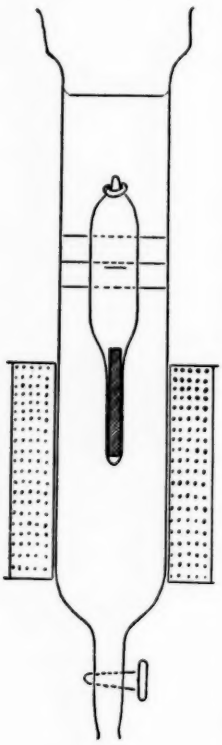

Fig. 1.

A third and indirect, very accurate, method is that of determining by titration the amount of chlorine contained in the seawater. This method is almost exclusively used by the international organization for the study of the North Sea. From the amount of chlorine in the water, its salinity as well as its density at different temperatures can be found from Knudsen's well-known hydrographic tables.

While I was engaged in certain investigations at Bornö Station, its director, Professor Otto Pettersson called my attention to the want of an improved method of total immersion more convenient than those described above, and as accurate as the chemical method. It then occurred to me that considerable advantage might be gained with a hydrometer of total immersion worked by electromagnetic compensation, *i. e.*, counterbalancing the buoyancy of a float completely immersed in seawater by an electromagnetic force acting on a piece of soft iron within the float.

The instrument constructed according to this idea is shown in Fig. 1. The float is drawn out into a narrow tube, that contains a rod of soft iron, 4 cm. in length, and some 3 mm. wide. The outer vessel, which contains the liquid to be investigated, has its lower part drawn out to the shape shown by the figure. It is surrounded by a coil, that has a length of about 12 cm. If the force acting on a piece of soft iron, which varies both with the intensity and the divergency of the E.M. field, is plotted against the vertical distance from the center of the coil (C), we would obtain a curve of the general type shown in Fig. 2. Suppose now that the electromagnetic force is counterbalanced by the buoyancy of the float, containing the piece of soft iron. The float will then come to rest in a position, where the E.M. force is equal to its buoyancy. The equilibrium is stable provided that the float is brought to rest at a place corresponding to a certain point (P) on the part AC of the curve. An increase

of the current through the coil or a decrease of the buoyancy of the float, viz., the specific gravity of the liquid, will bring the float nearer to the center (A). If the float is always brought to rest at a fixed distance from the center, a certain current through the coil will always correspond to the same buoyancy. The adjustment is very sensitive at points corresponding to the maximum of the curve, where dl/df has its largest value.

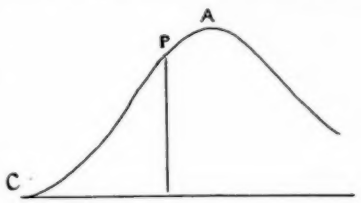

Fig. 2.

A determination is made in the following way. The water sample is poured into the vessel which has previously been washed out with another part of the same sample. The float is loaded with a fine platinum spiral of known weight until it very nearly sinks in the liquid. By sending a current through the coil the float is depressed until the mark etched on the float coincides with that on the wall of the envelop. The current is measured on a milliammeter and from the observed value the specific gravity of the water sample can be found from an empirical table obtained by standardizing the instrument once for all with a few samples of different salinities checked through titrations. The final adjustment is made by the E.M. compensation. The rough adjustment is realized by loading the float with a set of light platinum spirals of known weight. In this way the use of the instrument can be extended over the whole range of specific gravity occurring in oceanographic research, *i. e.*, 1.000 to 1.028, whereas the sensibility can nevertheless be kept as high as 0.00004 or even higher. This sensibility allows of measuring the salinity of a water sample with the degree of accuracy agreed upon as sufficient for modern research at the International Congress of Stockholm, 1899.

I may finally include here some observations made with a preliminary type of the described instrument. The load of the float was adjusted until the float was just nearly sinking in a water sample of a specific gravity corresponding to a salinity of about 23.4 pro mille. Varying the salinity, I must send currents of varying strength through the coil, in order to keep the float at the mark. With the aid of a current between 0 and 75 milliamperes, I may then control a range corresponding to a difference of salinity of about 2 pro mille. In Fig. 3 the relation between the current (y) and the corresponding salinity (x) is given by a curve. The observations are represented by circles. The permeability of the iron not being constant, the relation is evidently not a linear one. But if we confine ourselves to work with fields that are not too small, viz.,

with currents larger than 35 milliamperes for the special kind of instrument here used, the relation may practically be regarded as linear. It must however be noticed that the sensitiveness of the instrument is larger the less the current is and it may therefore sometimes be found favorable to work with small currents, which always can be attained

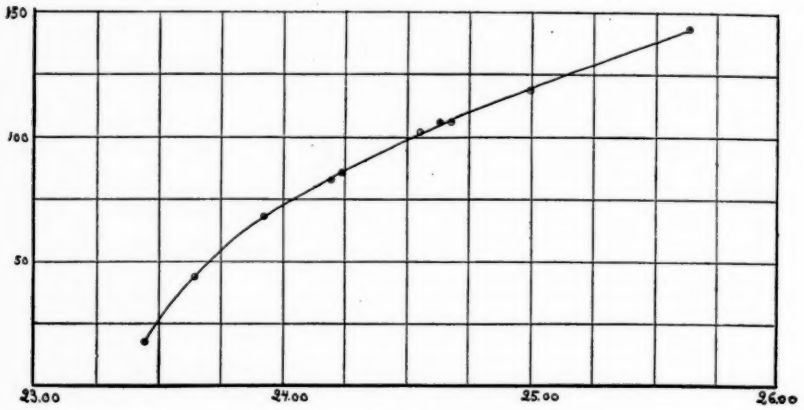

Fig. 3.

The Ordinates give currents used for compensation. The Abscissas represent satinity of water sample, given in pro mille.

through extending the use of loads. The mean deviation of the observations from the curve corresponds to less than 0.01 per cent. of salinity. The method may naturally be applied with advantage in all cases where the specific gravity of liquids is to be determined with high degree of accuracy, within relatively small intervals.

BORNÖ STATION, SWEDEN,
December, 1914.

SIGNAL PROPAGATION IN DISPERSIVE MEDIA.

BY WALTER COLBY.

IN a recent paper[1] Prof. Sommerfeld has discussed a problem originally suggested by the relativity theory, viz., Can a signal be transmitted in a dispersive medium with a velocity greater than that of light in vacuum. One is naturally led to this question when he considers that in the case of anomalous dispersion where the value of the index of refraction may be less than unity, the actual rate of progress of a given wave crest, in a monochromatic wave train does exceed its velocity in vacuum. The solution of the problem, however, does not follow straightway, for a signal must differ from a monochromatic wave train in that it has some fiducial mark, some irregularity in form, a beginning or an end by means of which one identifies it at the detector. And in this irregularity one necessarily introduces new frequencies. A monochromatic wave train cannot therefore be used for this problem. Professor Sommerfeld has taken as the simplest form of signal a sine wave of constant amplitude beginning at a given moment and persisting thereafter. The cessation of such a wave train would not differ from the superposition of a precisely similar disturbance in opposite phase. A more complicated signal could likewise be reduced by analysis to a series of simple waves of the above type and would therefore add nothing to the generality of the result. The rather surprising conclusion of the paper cited above is that although the rate of progress of the signal-front cannot exceed vacuum velocity even at an absorption band, it cannot fall below it in the remaining regions. To be sure, the amount of energy which reaches the new point at the first instant is too small to affect any known detector and to measure the signal velocity by it or to speak of this extremely small disturbance as the signal may be questionable. In the same number of the Annalen, L. Brillouin[2] has defined the arrival of the signal as that moment at which the disturbance reaches the order of magnitude of its final value. He also identifies signal velocity with group velocity except in the region of anomalous dispersion where this term loses its meaning.

[1] Ann. d. Phys., 44, p. 177, 1914.
[2] Ann. d. Phys., 44, p. 203, 1914.

The problem naturally extends itself to an investigation of the changes of form which such a "signal front" may experience at the boundary of a dispersive medium. As Professor Sommerfeld has pointed out, since rectilinear propagation acquires the characteristics of the medium only gradually, one may readily expect the wave to suffer a similar initial change on refraction and reflection.

In the present paper I have undertaken an investigation of the changes experienced by a signal at a plane boundary surface both with reference to its form and direction of propagation, also a brief discussion of the reflected wave. The method used by Sommerfeld has been adapted with very slight variation to the first part of this problem.

Let us consider the signal as existing only between the time limits $t = 0$ and $t = T$, and having the form

$$f(t) = 0 \qquad \text{for} \quad t < o,$$
$$f(t) = \sin\frac{2\pi t}{\tau} \quad \text{for} \quad o < t < T,$$
$$f(t) = 0 \qquad \text{for} \quad t > T.$$

Putting this in the form of a Fourier integral we have

$$f(t) = \frac{1}{\pi}\int_0^\infty dn \int_0^T \sin\frac{2\pi\alpha}{\tau}\cos n(t-\alpha)d\alpha$$

If T, the duration of the signal, be large in comparison with τ, the period of vibration of the incident light, we may without loss of generality set $T = N\tau$, where N is an integer. Performing the integration with respect to the variable α we obtain

$$f(t) = \frac{2}{\tau}\int_0^\infty \frac{dn}{n^2 - \left(\frac{2\pi}{\tau}\right)^2}(\cos n(t-T) - \cos nt).$$

Or in the exponential form

$$f(t) = \text{Real part of } \frac{1}{\tau}\int \frac{dn}{n^2 - \left(\frac{2\pi}{\tau}\right)^2}(e^{-in(t-T)} - e^{-int}).$$

The complex integral has the great advantage that the two terms may be treated separately and the points ($n = \pm 2\pi/\tau$) avoided by a deformation of the integration path in the complex n plane. In the real integral these points present convergence difficulties unless the whole integrand be treated together. Moreover a separation of the terms is highly desirable for, as we see by inspection, one represents the setting-in of the signal and the other its cessation. In fact, in the following work, it is only necessary to consider the "front" of the signal as is done in

Sommerfeld's paper, whereby our integral reduces to

$$(1) \qquad f(t) = \frac{1}{\tau}\int_{+\infty}^{-\infty} \frac{dn}{n^2 - \left(\frac{2\pi}{\tau}\right)^2} e^{-int},$$

where the integration path extends from $+\infty$ to $-\infty$ along the real axis with deformations about the poles $\pm (2\pi/\tau)$. Let us now consider the surface of separation of the vacuum and the dispersive medium as the plane $x = 0$. If the wave be incident at an angle θ, the integral in medium one ($x < 0$) has the form

$$(2) \qquad \frac{1}{\tau}\int_{+\infty}^{-\infty} \frac{dn}{n^2 - \left(\frac{2\pi}{\tau}\right)^2} e^{-in\left(t - \frac{x\cos\theta + y\sin\theta}{c}\right)}.$$

For the second medium the amplitude is altered with the help of the Fresnel formulas. If the above integral represents either component of the electric vector then we may use the factors

$$D_p = \frac{2}{\mu + \frac{\cos\varphi}{\cos\theta}} = \frac{2\mu\cos\theta}{\mu^2\cos\theta + \sqrt{\mu^2 - \sin^2\theta}},$$

$$D_s = \frac{2}{1 + \mu\frac{\cos\varphi}{\cos\theta}} = \frac{2\cos\theta}{\cos\theta + \sqrt{\mu^2 - \sin^2\theta}}$$

respectively for the two components, where

$$\mu^2 = 1 + \frac{a^2}{n_0^2 + 2i\rho n - n^2},$$

$$a^2 = \frac{Ne^2}{m}.$$

N, e, m, n_0 and ρ denoting respectively the number of resonators per c.c., their electric charge, mass, natural vibration frequency multiplied by 2π, and damping constant.

The corresponding alteration for the exponential factor may be deduced from the following considerations. Let us assume the form to be $e^{i(\alpha x + \beta y - nt)}$. Now since the loci of equal amplitudes are planes parallel to the surface $x = 0$, the coefficient of y must be $\beta = (n\sin\theta)/c$ as before. If we denote by φ the complex angle of refraction then $\alpha/\beta = \cot\varphi$, $\alpha = \beta\cot\varphi = n/c\sin\theta\cot\varphi$. Since $\mu = \sin\theta/\sin\varphi$, then $\sin\theta\cot\varphi = \sqrt{\mu^2 - \sin^2\theta}$, and $\alpha = n/c\sqrt{\mu^2 - \sin^2\theta}$. The total integral for the component of the electric vector normal to the plane of incidence, for $x > 0$, is therefore

$$(3)\quad f(t,x,y)=\frac{1}{\tau}\int\frac{2\cos\theta}{\cos\theta+\sqrt{\mu^2-\sin^2\theta}}\,e^{-in\left(t-\frac{x\sqrt{\mu^2-\sin^2\theta}+y\sin\theta}{c}\right)}\frac{dn}{n^2-\left(\frac{2\pi}{\tau}\right)^2}$$

Our problem now reduces itself to the evaluation of integral (3). If we consider the n complex plane we note again the poles on the real axis at $\pm 2\pi/\tau$, also branch cuts occasioned by the square root $\sqrt{\mu^2-\sin^2\theta}$. Locating the branch points by setting $\sqrt{\mu^2-\sin^2\theta}$ equal successively to 0 and ∞ we find them to lie at

$$U=-\rho\pm\sqrt{n_0^2-\rho^2+\frac{a^2}{1-\sin^2\theta}},$$

$$N=-i\rho\pm\sqrt{n_0^2-\rho^2}.$$

The branch cuts may therefore be drawn (Fig. 1) parallel to the real axis in the lower half plane and symmetrically placed with respect to the imaginary axis. The inner points are fixed by the properties of the medium but the outer ones depend also on the obliquity of the ray.

Let us turn again to the question of the integration path which, for the original integral, was along the real axis. One may convince himself that, for the exponential form here chosen, the deformation of the integration path about the real poles must be made in the upper half plane, by an integration of expression (1) to which (3) reduces for the point $x=y=0$. If the integration path be correctly chosen, the result of this integration should be the original form of disturbance assumed. Rewriting integral (1)

$$f(t)=\frac{1}{\tau}\int\frac{dn}{n^2-\left(\frac{2\pi}{\tau}\right)^2}e^{-int}.$$

We note that if we set $n=a+ib$, we have as a factor e^{bt} which causes the integrand to vanish for $t<o$ and b very large. Therefore if we deform the integration path upwards we see that the integral vanishes for $t<o$. Deforming downwards a similar result would be obtained for $t>o$, were it not for the circuits hanging about the poles $\pm(2\pi/\tau)$. These may be evaluated by Cauchy's theorem of residues with the result

$$\frac{2\pi i}{\tau}\left[\frac{e^{-int}}{n-\frac{2\pi}{\tau}}\right]_{n=-\frac{2\pi}{\tau}}+\frac{2\pi i}{\tau}\left[\frac{e^{-int}}{n+\frac{2\pi}{\tau}}\right]_{n=\frac{2\pi}{\tau}}=\frac{1}{2}\left(e^{-\frac{2\pi t}{\tau}}-e^{\frac{2\pi t}{\tau}}\right)=\sin\frac{2\pi t}{\tau}.$$

The result is therefore

$$f(t)=0 \qquad \text{for} \quad t<o.$$

$$f(t)=\sin\frac{2\pi t}{\tau} \qquad \text{for} \quad t>o \text{ as assumed.}$$

Turning now to Integral (3) and deforming the path upward to large positive imaginary values of n, μ becomes equal to unity and the integral simplifies to

$$\frac{I}{\tau}\int e^{-in\left(t-\frac{x\cos\theta+y\sin\theta}{c}\right)}\frac{dn}{n^2-\left(\frac{2\pi}{\tau}\right)^2}.$$

We have already found that this form of integral vanishes when the factor of $-in$ in the exponential part has a negative sign, that is, when

$$t<\frac{x\cos\theta+y\sin\theta}{c}.$$

Here we see as in Prof. Sommerfeld's paper that no disturbance can reach the point whose coördinates are x, y in less time than $(x\cos\theta+y\sin\theta)/c$, *i. e.*, more quickly than the light could traverse the intervening distance in vacuum. The possibility of more rapid propagation in the case of anomalous dispersion is thus excluded.

For values of t greater than the one defined above, the integral will vanish in the region where n has a large negative imaginary part. If we deform the path downwards, however, circuits are left about the poles and branch cuts. The position of the branch cuts in the complex plane leads us to expect in the result an exponential factor vanishing with increasing time whereas the integration about the real poles will be free from it. The former then will represent the free and the latter the forced vibrations. The integration about the branch cuts cannot be further evaluated but we may use Cauchy's theorem again for the calculation of the forced vibrations with the result

$$\frac{2\pi i}{\tau}\left\{\frac{2\cos\theta}{\cos\theta+\sqrt{\mu^2-\sin^2\theta}}\,\frac{e^{-in\left(t-\frac{x\sqrt{\mu^2-\sin^2\theta}+y\sin\theta}{c}\right)}}{n+\frac{2\pi}{\tau}}\right\}_{n=\frac{2\pi}{\tau}}$$

$$+\frac{2\pi i}{\tau}\left\{\frac{2\cos\theta}{\cos\theta+\sqrt{\mu^2-\sin^2\theta}}\,\frac{e^{-in\left(t-\frac{x\sqrt{\mu^2-\sin^2\theta}+y\sin\theta}{c}\right)}}{n-\frac{2\pi}{\tau}}\right\}_{n=-\frac{2\pi}{\tau}}.$$

Changing the sign of n in μ^2 only affects its imaginary part giving the conjugate value. If then we set $\sqrt{\mu_+^2-\sin^2\theta}=\alpha+i\beta$ and $\sqrt{\mu_-^2-\sin^2\theta}=\alpha-i\beta$ where μ_+^2, μ_-^2 are the values taken by μ^2 when $n=\pm 2\pi/\tau$ respectively, the two terms in our result may be combined. Taking then the real part we have

$$\frac{2\cos}{(\alpha+\cos\theta)+\beta^2}e^{-\frac{2\pi}{c\tau}\beta x}\left\{(\alpha+\cos\theta)\sin\frac{2\pi}{\tau}\left(t-\frac{\alpha x}{c}-\frac{y\sin\theta}{c}\right)\right.$$
$$\left.+\beta\cos\frac{2\pi}{\tau}\left(t-\frac{\alpha x}{c}\frac{y\sin\theta}{c}\right)\right\}.$$

The exponential factor contains x but not t. This is therefore a part of the whole disturbance which does not decrease as time goes on but whose amplitude diminishes the deeper it penetrates the second medium. For large values of t it represents the complete disturbance. A calculation of the forced vibration for the component of the electric field parallel to the plane of incidence gives a similar expression. The change of phase at the boundary as indicated by the coefficients of the sine and cosine terms is a different one for the two components, showing the familiar change in state of polarization on refraction.

For an investigation of the integral for smaller values of t, we may proceed most simply by restricting ourselves to regions not in the neighborhood of an absorption band, whereby $2\pi/\tau \neq n_0$ and ρ may be taken vanishingly small. This enables us to develop the expression $\sqrt{\mu^2-\sin^2\theta}$ as a power series and integrate straightforwardly. We may very conveniently change the variable by setting

$$n = m\frac{2\pi}{\tau} \quad \text{and} \quad n_0 = m_0\frac{2\pi}{\tau}.$$

Then

$$\sqrt{\mu^2-\sin^2\theta} = \sqrt{1+\frac{a^2}{m_0^2-m^2}\left(\frac{\tau}{2\pi}\right)^2-\sin^2\theta}$$

and

$$dn = dm\cdot\frac{2\pi}{\tau}.$$

If we also write

$$K = m\frac{2\pi}{c\tau}\left\{\sqrt{1+\frac{a^2}{m_0^2-m^2}\left(\frac{\tau}{2\pi}\right)^2-\sin^2\theta}-\cos\theta\right\}$$

and

$$t' = t-\frac{x\cos\theta+y\sin\theta}{c}.$$

Integral (3) becomes

$$(4)\qquad \frac{1}{2\pi}\int_{\infty}^{-\infty}\frac{2dm}{1+\sqrt{1+\frac{a^2}{m_0^2-m^2}\left(\frac{\tau}{2\pi}\right)^2-\sin^2\theta}}\frac{e^{ikx-im\frac{2\pi t'}{\tau}}}{m^2-1}.$$

Let us now develop the integrand as a descending power series valid for large values of m.

$$\left\{1-\left[\frac{a^2}{m^2-m_0^2}\frac{\tau^2}{4\pi^2}+\sin^2\theta\right]\right\}^{\frac{1}{2}}.$$

may be put in the form

$$\cos\theta - \frac{a^2\tau^2}{8\pi^2}\left\{\frac{\varphi_1}{m^2} + \frac{\varphi_2}{m^4} + \cdots\right\},$$

where φ_n is a polynomial easily evaluated for the lower orders.

$$Kx = -\frac{a^2\tau x}{4\pi c}\left\{\frac{\varphi_1}{m} + \frac{\varphi_2}{m^3} + \frac{\varphi_3}{m^5} + \cdots\right\} = -\xi\left\{\frac{\varphi_1}{m} + \frac{\varphi_2}{m^3} + \cdots\right\}$$

where

$$\xi = \frac{a^2\tau x}{4\pi c}.$$

Substituting in $e^{ikx}/(m^2 - 1)$ and collecting terms we have

$$\sum_{\nu=0}^{\infty} \frac{1}{m^{\nu+2}} \frac{(-i\varphi_1\xi)^\nu}{\nu!} + \left\{1 + \varphi_1\left(\frac{-i\xi}{m}\right)\right\} \sum_0^\infty \frac{1}{m^{\nu+4}} \frac{(-i\xi\varphi_1)^\nu}{\nu!} + \cdots$$

The amplitude $(2\cos\theta)/(\cos\theta + \sqrt{\mu^2 - \sin^2\theta})$ gives when developed

$$1 + \frac{\alpha^2}{4}\frac{1}{m^2} + \left\{\frac{\alpha^2}{8} + \frac{\alpha}{4}m_0^2\right\}\frac{1}{m^4} + \cdots$$

where

$$\alpha = \frac{a^2\tau^2}{4\pi^2\cos^2\theta}.$$

The total integrand may then be written

$$\left[\Sigma \frac{1}{m^{\nu+2}} \frac{(-i\xi\varphi_1)^\nu}{\nu!} + \left\{\frac{\alpha}{4} + 1 + \frac{(-i\xi\varphi_1)}{m}\right\} \Sigma \frac{1}{m^{\nu+4}} \frac{(-i\xi\varphi_1)^\nu}{\nu!} + \cdots\right] e^{-im\frac{2\pi t'}{\tau}}$$

The singularities consist now of poles at the origin. Since the integral still vanishes where m has a large negative imaginary part we may replace our integration path across the upper half plane by a circle of infinite radius with its center at the origin. Applying again the Cauchy theorem we obtain for the first term

$$\frac{1}{2\pi}\sum_0^\infty \left(\frac{-i\xi\varphi_1}{\nu!}\right)^\nu \int e^{-im\frac{2\pi t'}{\tau}} \frac{dm}{m^{\nu+2}} = \frac{1}{2\pi}\sum_0^\infty \left(\frac{-i\xi\varphi_1}{\nu!}\right)^\nu \frac{2\pi i}{(\nu+1)!}\left(\frac{-2\pi i t'}{\tau}\right)^{\nu+1}$$

$$= \frac{2\pi t'}{\tau}\sum_0^\infty \frac{\left(\frac{-2\pi\xi t'\varphi_1}{\tau}\right)}{\nu!\,(\nu+1)!}.$$

By comparison with the general expression for Bessel's function of the nth order

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_0^\infty \frac{\left(\frac{-x^2}{4}\right)^\nu}{\nu!\,(\nu+n)!}.$$

We may write the above result

$$\sqrt{\frac{2\pi t'}{\tau\xi\varphi_1}}\, J_1\left(2\sqrt{\frac{2\pi\xi\varphi_1 t'}{\tau}}\right).$$

Later terms give Bessel's functions of higher orders but with the same argument. Omitting the arguments we have for the complete result

$$\sqrt{\frac{2\pi t'}{\tau\xi\varphi_1}}J_1 - \left(\frac{\alpha}{4}+1\right)\left(\frac{2\pi t'}{\tau\xi\varphi_1}\right)^{\frac{3}{2}}J_3 + \varphi_1\xi\left(\frac{2\pi t'}{\tau\xi\varphi_1}\right)^{\frac{4}{2}}J_4 + \cdots.$$

For very small values of t' the first term exceeds the others in order of magnitude. This is assured not only by the increasing order of t'/ξ but also by the Bessel's functions themselves which vanish at the origin as powers which grow with the order. It will be noted that boundary effects are limited to those terms which contain α. If therefore we set $\theta = 0$ in the first term it should give us Sommerfeld's result. $\varphi_1 = 1/\cos\theta = 1$ for $\theta = 0$. We have therefore for the first forerunners in the case of normal incidence

$$\sqrt{\frac{2\pi t'}{\tau\xi}}J_1\left(2\sqrt{\frac{2\pi\xi t'}{\tau}}\right).$$

This form of the result has been fully discussed in the Sommerfeld paper and plotted as an approximate sine wave with increasing amplitude and period. Obliquity at the boundary has therefore an effect of decreasing the amplitude and period of these early forerunners. The next term in the series to appear with increasing t' is the third (due to the factor ξ). If we use as numerical values of the constants

$$m_0 = 10,\ \tau = 1.6\cdot 10^{-15},\ \xi = 7.8\cdot 10^{6}$$

also

$$\theta = 0,\ \varphi_1 = 1,\ \alpha = 125$$

whereby the time between the first two maxima is $4\cdot 10^{-22}$ sec. we find the third term of our series attains the magnitude of the first when t' has the order of magnitude of 10^{-14}, that is, many periods after the theoretical beginning of the disturbance.

Direction of The Ray.

For the study of the deviation of the signal ray it is much more convenient to think of the total electric vector of the incident beam as parallel to one of the coördinate axes. Let us again take the plane $z = 0$ as the plane of incidence and consider the electric vector as perpendicular to it. Integral (3) may again be taken to represent this quantity but for brevity we shall denote it by the symbol

$$\int \Phi dn = Z.$$

For the other two components we have necessarily

$$X = Y = 0.$$

To deduce the values of the components of the magnetic vector, we may use the Maxwell equations

$$-\frac{1}{c}\frac{\partial H}{\partial t} = \text{curl } E.$$

Denoting the x component by α, we have

$$-\frac{1}{c}\frac{\partial \alpha}{\partial t} = \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \quad \frac{\partial \alpha}{\partial t} = -\frac{1}{\tau}\int in \sin\theta \Phi dn$$

$$\alpha = -\frac{1}{\tau}\int\int in \sin\theta \Phi dn\, dt = \frac{1}{\tau}\sin\theta \int \Phi dn.$$

Similarly

$$\beta = -\frac{1}{\tau}\int \sqrt{\mu^2 - \sin^2\theta}\,\Phi dn$$

$$\gamma = 0.$$

We are now ready to build up the vector product of the electric and magnetic fields and form the Poynting vector. For the three components we obtain

$$S_x = \frac{c}{4\pi}(\gamma Y - \beta Z) = \frac{c}{4\pi\tau^2}\int \sqrt{\mu^2 - \sin^2\theta}\,\Phi dn \int \Phi dn,$$

$$S_y = \frac{c}{4\pi}(\alpha Z - \gamma X) = \frac{c}{4\pi\tau^2}\sin\theta \int \Phi dn \int \Phi dn,$$

$$S_z = \frac{c}{4\pi}(\beta X - \alpha Y) = 0.$$

If we now denote the angle of refraction by δ we may write

$$\cot\delta = \frac{S_x}{S_y} = \frac{\int \sqrt{\mu^2 - \sin^2\theta}\,\Phi dn}{\sin\theta \int \Phi dn}.$$

We are interested here only in the initial direction of the ray and must seek an evaluation of this expression for values of t' slightly greater than zero. Obviously the method last employed will serve here also. We have already obtained developments for the factors in the integrand of the numerator. Multiplying we obtain

$$\sqrt{\mu^2 - \sin^2\theta}\,\Phi dn = \cos\theta \int \Phi dn - \frac{1}{2\pi}\int \frac{a^2\tau^2\varphi_1}{8\pi^2} e^{-im\frac{2\pi t'}{\tau}} \sum_0^\infty \frac{1}{m^{\nu+4}} \frac{(-i\xi\varphi_1)^\nu}{\nu!} + \cdots.$$

Integrating as before

$$= \cos\theta \int \Phi dn + \frac{a^2\tau^2}{8\pi^2}\varphi_1 \left(\frac{2\pi t'}{\tau\xi\varphi_1}\right)^{\frac{3}{2}} J_3\left(2\sqrt{\frac{2\pi\xi\varphi_1 t'}{\tau}}\right) + \cdots.$$

Here again we have a series which for small values of t' does not differ in value from its first term. Dividing by $\sin\theta\int\Phi dn$ we find likewise for vanishingly small values of t'

$$\cot\delta = \frac{\cos\theta}{\sin\theta} = \cot\theta, \quad \delta = \theta.$$

In other words *the initial front of the wave train which penetrates the dispersive medium with the velocity of light in vacuum, suffers no deviation at the boundary.* As t' grows we must consider later terms in the series solution. Suppose for a first approximation we neglect terms later than the first in the series solution of the denominator. Dividing the complete numerator we then obtain

$$\cot\delta = \cot\theta + \frac{ct'}{x\sin\theta}\frac{J_3}{J_1} + \frac{8\pi^2c^2t'^2}{a^2\tau^2x^2\varphi_1^2\sin\theta}\left(\frac{\alpha^2}{4}\varphi_1 + \varphi_1 + \varphi_2\right)\frac{J_5}{J_1} + \cdots.$$

The ratio J_n/J_1 equals zero for zero argument ($n > 1$) and grows slowly at first with increasing argument. Its value also decreases with rising order of the numerator for small values of the argument. So the ray

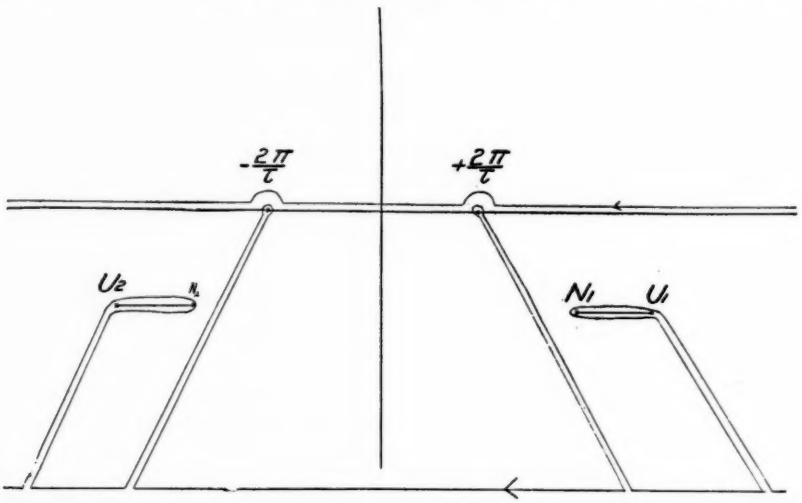

Fig. 1.

suffers no deviation until t' attains a sufficiently high value for the second term to be comparable with the first. All factors of this term are essentially positive for values of the argument less than 3.83 +, the value of the first root of J_1. Cot δ will therefore increase from the original value cot θ with increasing t', *i. e.*, δ will become smaller, the ray will be bent continuously toward the normal. The terms succeeding the first appear here much more rapidly then in the series just discussed as one may see

by calculating the magnitude of the factor of $t' J_3/J_1$. For $\sin\theta = 1/10$, $x = 1$ it becomes 3.10^{11}. The factor of $t' J_5/J_1$ for the same angle rises to the magnitude of 10^{22}. That these terms do appear in the above mentioned order is assured by the increasing power of t' and the increasing order of J_n in the numerator.

THE REFLECTED RAY.

To build up an integral for the reflected ray is a much simpler task. The exponential part retains the form of integral (2) except for sign. There remains only to alter the amplitude by the Fresnel formula and we have for the component normal to the plane of incidence

$$(5) \qquad \frac{1}{\tau}\int \frac{\cos\theta - \sqrt{\mu^2 - \sin^2\theta}}{\cos\theta + \sqrt{\mu^2 - \sin^2\theta}} e^{-in\left\{t + \frac{(x\cos\theta - y\sin\theta)}{c}\right\}} \frac{dn}{n^2 - \left(\frac{2\pi}{\tau}\right)^2}.$$

This integral has the same poles and branch cuts as (3) and we could in the same way deduce the expression for the forced vibration. As the

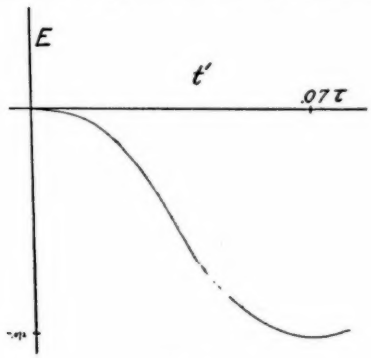

Fig. 2.

operation does not differ from that in the early part of the paper and the result is so well known we may easily omit it. Our interest here centers, as it did in the preceding discussions, on the behavior of the wave immediately after its arrival. We shall proceed therefore as before to a development of the integrand as a power series. The same change of variable is advantageous here and we may set

$$t' = t + \frac{x\cos\theta - y\sin\theta}{c}.$$

We have already obtained a development of $\sqrt{\mu^2 - \sin^2\theta}$. Substituting we obtain

$$\frac{1}{\tau}\int \left\{ \frac{\varphi_1\beta}{m^4} + \frac{(\varphi_1 + \varphi_2)\beta + \varphi_1^2\beta^2}{m^6} + \cdots \right\} e^{-\frac{2\pi i m t'}{\tau}} dm,$$

where

$$\beta = \frac{a^2\tau^2}{16\pi^2 \cos\theta}.$$

Developing the exponential into an ascending series, multiplying into the above and choosing only those terms of power -1 in m which will therefore give residues on integration about the pole $m = 0$, we have

$$-\varphi_1\beta\frac{\left(\frac{2\pi}{\tau}\right)^3}{3!}t'^3 + [(\varphi_1+\varphi_2)\beta + \varphi_1^2\beta^2]\frac{\left(\frac{2\pi}{\tau}\right)^5}{5!}t'^5 - \cdots.$$

This series suggests a sine development with increasing coefficients and first term gone. The order of magnitude of the coefficients is determined by the term containing φ_n with the highest subscript. The order of magnitude of φ_n is that of $(m_0^2)^{n-1}$. $\beta = a^2\tau^2/16\pi^2 = 31$ for $\theta = 0$, a and τ chosen as before. An approximate calculation of the coefficients may easily be made for several terms. The first minimum lines at $2\pi t'/\tau = .45$, $t' = .07\,\tau$. Its ordinate has a value of about .072. The amplitude of the forced vibration in the reflected ray for the same numerical constants, has the value .2. Moreover since t' appears in no lower power than the cube the curve is tangent to the t' axis at the origin (Fig. 2). We see therefore just as we did in the case of refraction that the signal front is of high frequency and greatly robbed of energy. Both facts are naturally closely connected with the phenomenon of setting the resonators in vibration and it is only after they have reached their final states of motion that the refracted ray swings into its final deviated position and both waves acquire their maximum amplitudes.

SUMMARY.

By integrating the Fourier integral in the complex plane, a study has been made of the progress of a light "signal" incident obliquely at the plane boundary of a dispersive medium.

1. The forced vibration of the refracted wave has been deduced with the familiar changes in amplitude, phase and state of polarization.

2. An expression has been found for the refracted "forerunners" reducible to the form given in Sommerfeld's paper for normal incidence and small values of time. The forerunners are found in general not to differ from the type described by Sommerfeld except in magnitude of amplitude and period, both quantities decreasing with increasing obliquity.

3. The direction of the refracted ray is found to be a function of the time, varying from the incident direction at the first instant continuously toward the normal.

4. The reflected ray has also a train of forerunners of extremely small amplitude and period. The curve describing this disturbance is however tangent to the time axis at the origin and rises more slowly than the refracted wave to its first maximum.

In addition, I might point out how remarkably this type of integration adapts itself to such problems. The algebra of complex numbers is already indispensable in modern optics and complex integration will not fall behind it in usefulness.

In conclusion I wish to thank Professor Sommerfeld for his kind assistance and invaluable suggestions throughout the progress of this paper.

PROCEEDINGS

OF THE

AMERICAN PHYSICAL SOCIETY.

ON THE VARIATION IN THE SENSITIVITY OF MOVING COIL GALVANOMETERS.[1]

BY PAUL E. KLOPSTEG.

DIFFERENCES in throws on the two sides of the null position of a moving coil galvanometer used ballistically result almost entirely from non-uniformity of the so-called radial field, and from slight displacements of the coil from its position of symmetry with the field. For a given quantity of electricity, discharged in opposite directions, a certain angular position with reference to the "geometric null" may be found for which equal throws are obtained. This position is not fixed for different quantities. The logarithmic decrement usually increases with the amplitude, and the variation is not the same on the two sides of the null position. Consequently the ballistic constant may have a different value for each value of the ballistic throw over the entire scale. In a supposedly high grade instrument differences as high as 20 per cent. in the constant have been found.

In a given deflection or ballistic instrument the best condition for proportionality of deflections and throws with steady and instantaneous currents, respectively, obtains when the axis of the coil is coincident with the axis of symmetry of the magnet. To keep these axes coincident when the coil assumes different angular positions it is necessary that the upper suspension terminals be short—so as to avoid the possibility of bent terminals—and parallel to the fiber, and that the lower suspension be so constructed as to exert no lateral force upon the coil. This last named condition is closely complied with if the torque is made small, and if there is no deformation in the regular turns of the spiral. The adjustment of the position of the coil in the field when once carefully made should be easily reproducible when disturbed. A very convenient method is to provide the galvanometer with an adjustable leveling device such as a spirit level, or a plumb-line with a movable indicator. The level or indicator may be set permanently when the desired adjustment of the coil is accomplished.

[1] Abstract of a paper presented at the Philadelphia meeting of the Physical Society, December 31, 1914.

The bifilar action of a twisted strip upper suspension has been found negligible in its effect upon galvanometer sensitivity. The effect of difference in torques for twisting and untwisting of the lower suspension (1.5-mil strip, 7 turns) has been found comparable with experimental error.

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PRELIMINARY NOTE ON A MERCURY VAPOR TUBE OSCILLATOR.[1]

BY B. LIEBOWITZ.

A METHOD for obtaining electrical oscillations of frequencies suitable for radio-telegraphy is described. A mercury vapor tube is provided with two mercury cathodes and an iron or graphite anode. Each cathode is connected through a large inductance and resistance to the negative terminal of a high voltage D.C. generator, and the anode, likewise through an inductance, to the positive terminal. An oscillatory circuit is connected between the cathodes. By this means oscillatory currents as large as ten amperes (r.m.s. value) at frequencies of the order of half a million have been obtained, but these oscillations were not steady. Very steady oscillations have been obtained at frequencies of about 35,000. This method seems to be specially adapted for high powers. The experiments are being continued.

It is pointed out that in all oscillators supplied with energy from a constant potential source, at least 50 per cent. of the energy must be wasted in the supply circuit.

COLUMBIA UNIVERSITY.

A NEW METHOD OF OBTAINING A HYSTERESIS LOOP.[1]

BY W. N. FENNINGER.

FOR students beginning the study of motors there had been designed an electromagnet to show that the force on a conductor placed in a magnetic field is given by the well-known expression $F = I'H'L'/10$, when the proper units are used. The pole faces of this magnet were accordingly made long and narrow (about 15 cm. by 4 cm.). Between the poles was placed a wire, each end of which dipped into a mercury cup. This wire was suspended by two dial spring balances, which could be raised or lowered easily so as to keep the wire always at the same height. It occurred to the author that this apparatus could be used to obtain a hysteresis loop. Since N, the number of turns on the coils of the electromagnet, and L, the length of the coils, were known, the magnetizing force can be computed from the usual expression $H = 4\pi NI/10L$. If we take the expression given above for the

[1] Abstract of a paper presented at the Philadelphia meeting of the Physical Society, December 31, 1914.

force on a conductor, and solve it for H', the intensity of the field, we get $H' = 10F/I'L'$. A current of 100 amperes in the conductor gave a maximum pull of a little more than 20 ounces on each balance. Tenths of an ounce were read on the balances. The current in the field was varied by small steps so as to have the iron pass through the hysteresis cycle, and a hysteresis loop was plotted from the results. The points fell on a smooth, closed curve, excepting two which deviated less than one per cent. The points in favor of this method are that there is no need of demagnetization to obtain the loop, that each determination of the flux is independent of every other determination and that the deflections to be read are steady ones and not momentary throws. One special advantage is the facility with which the closing of the loop may be checked. Aside from the simplicity of mechanical manipulation, the method has certain pedagogical advantages.

PRATT INSTITUTE, BROOKLYN,
Dec., 1914.

VAPORS WITH POSITIVE SPECIFIC HEAT IN ENERGY CONVERSION.[1]

BY J. E. SIEBEL.

IN this paper the author finds that the rate of energy conversion in an ideal complete reversible cycle, operated with a saturated vapor possessing a positive specific heat, like ether, is less than the maximal rate of conversion in a similar cycle operated with superheated vapor of ether.

The latter cycle follows the Carnot rate of efficiency while the efficiency of the saturated vapor cycle is expressed by

$$W = \frac{Q(t - t_0)}{T} - \frac{Q_1(t - t_0)}{T},$$

in which formula W represents the work obtained in reversible conversion between the temperatures t and t_0 when Q signifies the initial heat of vaporization at t degrees and Q_1 the heat which must be abstracted in the expansion stage in order to keep the vapor in a saturated condition. T represents the absolute temperature equal to "t" degrees common temperature.

The above formula as will be seen is a perfect analogon to the formula given in a former paper (abstracted in PHYSICAL REVIEW, April, 1914) for energy conversion with saturated vapors having a negative specific heat like steam, the only difference being that the + sign in the right hand side of the equation is replaced by a — (minus) sign in the equation applying to vapors with positive specific heat.

This difference is explained by the fact that in the former case heat is added by condensation of part of the vapor, while in the latter case heat is abstracted by evaporation of liquid introduced in the expansion stage in addition to the vapor.

CHICAGO, January, 1915.

[1] Abstract of a paper presented at the Philadelphia meeting of the Physical Society, December 31, 1914.